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Almost all hyperharmonic numbers are not integers

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A R T I C L E   I N F O

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A B S T R A C T

It is an open question asked by Mezö that there is no hyperharmonic integer except 1. So far it has been proved that all hyperharmonic numbers are not integers up to order $r = 25$. In this paper, we extend the current results for large orders. Our method will be based on three different approaches, namely analytic, combinatorial and algebraic. From analytic point of view, by exploiting primes in short intervals we prove that almost all hyperharmonic numbers are not integers. Then using combinatorial techniques, we show that if n is even or a prime power, or r is odd then the corresponding hyperharmonic number is not integer. Finally as algebraic methods, we relate the integerness property of hyperharmonic numbers with solutions of some polynomials in finite fields.

1. Introduction

The goal of this paper is to analyse the integerness property of hyperharmonic numbers. We answer Mezö's question [18] in almost all cases, which states that all hy-

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perharmonic numbers are not integers except 1. In the same paper, he also proved that there are none of them of order $r = 2, 3$. In [5,6], this was extended to $r \leq 25$ and they gave a set of integer pairs (n, r) such that $h_n^{(r)}$ is not an integer. In the current paper, we extend all these results in the literature applying analytic, combinatorial and algebraic tools.

Harmonic numbers are defined as the sequence of partial sums of the harmonic series

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

for $n \geq 1$. These numbers have been studied extensively and they are equipped with a lot of combinatorial and analytic properties. As a combinatorial one, it was proved that there is no harmonic number which is an integer except 1 [21]. This can also be seen by Bertrand's postulate, which was reproved by Erdös [14] from combinatorial point of view. Although almost all harmonic numbers are not integers, Wolstenholme [22] proved that $h_{p-1} \equiv 0 \bmod p^2$ for all primes $p \geq 5$. More recently this result was extended by Alkan in [1]. On the analytical side, Euler's harmonic zeta function which is defined by

$$\zeta_h(s) = \sum_{n=1}^{\infty} \frac{h_n}{n^s}$$

for $\Re(s) > 1$ satisfies the well-known striking relation (see [10, p. 252])

$$2\zeta_h(m) = (m+2)\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1) \tag{1.1}$$

for all $m \geq 2$, where the sum is zero when $m = 2$ and $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$. Euler's formula yields some important evaluations

$$\sum_{n=1}^{\infty} \frac{h_n}{n^2} = 2\zeta(3), \quad \sum_{n=1}^{\infty} \frac{h_n}{n^3} = \frac{5}{4}\zeta(4) = \frac{\pi^4}{72}.$$

Recently, by making use of connections to log-sine integrals which have remarkable applications to physical problems via potential energy of charged particle systems on the unit circle, Alkan (see [2] and [3]) showed that real numbers can be strongly approximated by combinations of values of $\zeta_h(s)$ and $\zeta(s)$, which resembles the classical result of Liouville's theorem on Diophantine approximation.

Hyperharmonic numbers were first introduced by Conway and Guy [11]. They are indeed a generalization of harmonic numbers. More precisely, the *n-th hyperharmonic number of order r* is defined recursively by

$$h_n^{(r)} := \sum_{k=1}^{n} h_k^{(r-1)},$$

where $h_n^{(1)} = h_n$. Hyperharmonic numbers also enjoy a lot of combinatorial properties. For instance a combinatorial approach to $h_n^{(r)}$ was given in [9] where they gave new expressions for $h_n^{(r)}$. A connection between hyperharmonic numbers and r-Stirling numbers, and new identities were obtained in [19]. It is known by [11] that $h_n^{(r)}$ can be expressed in terms of binomial coefficients and harmonic numbers with the formula

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}). \tag{1.2}$$

Many proofs related to the distribution of primes are based on the connection between combinatorial and analytic methods at the same time, such as sieve methods. In this paper, we also apply this interaction and we give a huge class where the corresponding hyperharmonic number is not an integer. Our first result is based on distribution of primes and it is a density estimate for non-integer hyperharmonic numbers. To do so, we turn our attention to the distribution of prime numbers. Let $\pi(x)$ be the number of prime numbers that are less than or equal to x. The prime number theorem, which was first proved in 1896 independently by Hadamard and de la Valée Poussin, states that

$$\pi(x) \sim \frac{x}{\log x}.$$

Using prime number theorem, one can prove that for any positive $\varepsilon > 0$ there exists a real number $x_0(\varepsilon)$ depending on ε such that if $x \geq x_0$ then the interval $[(1-\varepsilon)x, x]$ contains a prime number. The general question about primes in short intervals is the following: for which function $\Phi(x) = o(x)$ the interval $[x - \Phi(x), x]$ contains a prime number for large x depending on the function Φ? The current unconditional result is $\Phi(x) = x^{0.525}$ which was proved in [8]. Note that the Riemann hypothesis is equivalent to

$$\pi(x) = Li(x) + \mathcal{O}_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}\right)$$

where

$$Li(x) = \int_2^x \frac{1}{\log t}$$

is the logarithmic integral. Conditionally, assuming the Riemann hypothesis one can choose $\Phi(x)$ of size $x^{\frac{1}{2}+\varepsilon}$ for any positive ε. There is a far-reaching conjecture by Cramér, which is called the Cramér's model and it claims that $\Phi(x) = c\log^2 x$, for some $c > 0$. For more on Cramér's model and its analog in algebraic number theory, we direct the reader to [4].

Now we will state our first theorem and it states that almost all hyperharmonic numbers are not integers. More precisely, we will give a quantitative estimate for the number of pairs (n, r) lying in a rectangle where the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. To achieve this, we need to wait for large values of n in order to catch primes in short intervals. For the following theorem, see Proposition 9, Theorem 10 and Theorem 11:

Theorem 1.

(1) *Let* $S(x) = \left|\left\{(n, r) \in [0, x] \times [0, x] \,\middle|\, h_n^{(r)} \notin \mathbb{Z}\right\}\right|$. *In other words, $S(x)$ counts the number of pairs (n, r) in the rectangle $[0, x] \times [0, x]$ where the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Then we have* $S(x) = x^2 + \mathcal{O}\left(x^{\frac{2.475}{1.475}}\right)$, *which means that non-integer hyperharmonics have the full asymptotic in the first quadruple. Conditionally, assuming the Riemann hypothesis or Cramér's conjecture, the error term can be taken of size* $\mathcal{O}_\varepsilon\left(x^{\frac{5}{3}+\varepsilon}\right)$ *and* $\mathcal{O}\left(x^{\frac{3}{2}} \cdot \log x\right)$ *respectively.*
(2) *For a given natural number d, there exists a natural number n_d such that if $n \geq n_d$ and $r \leq dn$ then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Moreover there exists a natural number n_0 such that if $n \geq n_0$ and* $r \leq \frac{n^{1.475}-7n}{2}$ *then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. In particular for a fixed r, there are only finitely many possible values of n where $h_n^{(r)}$ may be an integer.*
(3) *For a fixed $n > 1$, there are infinitely many values of r such that $h_n^{(r)} \notin \mathbb{N}$.*

Our next theorem is of combinatorial flavour and it is based on the p-ary representations of n and r for a certain prime p. Also the proof uses the connection between the p-adic valuation of binomial coefficients and counting carries. Let I be a set and p be a prime. We define $\mu_p(I) := \max\{\nu_p(s) \,|\, s \in I\}$ as *the maximal power of p in I*. We also put $I(n, r) := \{r, ..., n+r-1\}$. For the following result, see Theorem 18 and Theorem 21:

Theorem 2. *If one of the following conditions (1)–(3) on n and r is satisfied, then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer:*

(1) *n is even.*
(2) *r is odd.*
(3) *n is a prime power.*

Now let a be nonnegative and m, k, b be positive integers. Let p be a prime number such that $a + b \leq p$. Put

$$n = mp + a = (n_\alpha, \dots, n_0)_p$$

$$r = kp + b = (r_\beta, \dots, r_0)_p,$$

where $(n_\alpha, \dots, n_0)_p$ and $(r_\beta, \dots, r_0)_p$ are the p-ary representations of n and r, respectively. If there exist $c \in \mathbb{N}$ and $\kappa > \alpha$ such that $r \in (cp^\kappa - n, cp^\kappa]$, then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Otherwise the non-integerness property still holds, if p is sufficiently large depending on m and k.

Our third theorem is on the algebraic side and it connects the non-integerness property of hyperharmonic numbers with solutions of certain polynomials in finite fields (see Theorem 23).

Theorem 3. *Let $n = kp^\alpha$ be an odd integer where p is a prime, $\alpha \geq 1$ and r is given. Define $a = \frac{k-1}{2}$, $c = \left\lceil \frac{r}{p^\alpha} \right\rceil$ and*

$$F_k(x) := \sum_{i=-a}^{a} \left[\prod_{j=-a}^{a} (x - j) \right] \frac{1}{x - i}.$$

Assume that $\nu_p(F_k(c+a)) \leq \nu_p(k!)$. Then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. In particular if $c+a$ is not a root of $F_k(x)$ in modulo p, then $h_n^{(r)} \notin \mathbb{N}$.

Our final theorem is a computational aspect of our methods and it is an application of the previous three theorems (see Corollary 24 and Corollary 26):

Theorem 4. *If one of the following conditions on n and r is satisfied, then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer:*

(1) $r \leq 16n$.
(2) $1 < n \leq 32$.
(3) $r \leq 20.001$.
(4) $n = 3p^\alpha$ *where* $\alpha \geq 1$, p *is a prime and* $p \equiv \pm 5 \bmod 12$.
(5) $n = 5p^\alpha$ *where* $\alpha \geq 1$, p *is a prime and*

$$p \equiv 7, 11, 13, 14, 19, 21, 22, 23, 26, 28, 31, 33, 38, 39, 41, 42, 44, 46, 52, 53, 56, \quad \bmod 145.$$
$$57, 61, 62, 63, 66, 67, 69, 76, 78, 79, 82, 83, 84, 88, 89, 92, 93, 99, 101, 103,$$
$$104, 106, 107, 112, 114, 117, 119, 122, 123, 124, 126, 131, 132, 134, 138$$

Throughout the paper, the numbers n, r will be always natural numbers. The set $\mathbb{P}$ denotes the set of prime numbers. We denote p as a prime in the entire paper. Also we use the p-adic valuation of a natural number in the usual way, i.e. for a given $a \in \mathbb{Z}$ we denote

$$\nu_p(a) := \begin{cases} m & \text{if } p^m \parallel a \\ \infty & \text{if } a = 0 \end{cases}$$

as the *p-adic valuation* (or the *p-adic order*) of a. Here $p^m \parallel a$ means $p^m \mid a$ but $p^{m+1} \nmid a$. We extend this notation to a rational number $q = a/b \in \mathbb{Q}$ by setting $\nu_p(q) = \nu_p(a) - \nu_p(b)$ where $a, b \in \mathbb{Z}$.

Now we will make a crucial observation which we will use subsequently in the whole paper for the non-integerness property. In order to show the non-integerness property of hyperharmonic numbers, it is enough to check the following: from (1.2), we have

$$\begin{aligned} h_n^{(r)} &= \frac{\mathrm{Per}(n+r-1, r-1)}{n!}\left(\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n+r-1}\right) \\ &= \frac{\mathrm{P}}{n!} \cdot \frac{\sum_{i=r}^{n+r-1} \frac{\mathrm{P}}{i}}{\mathrm{P}} \\ &= \frac{\sum_{i=r}^{n+r-1} \mathrm{P}_i}{n!}, \end{aligned} \tag{1.3}$$

where

$$\mathrm{P} = \mathrm{Per}(n+r-1, r-1) = \frac{(n+r-1)!}{(r-1)!} = r \cdot (r+1) \cdots (n+r-1)$$

and $\mathrm{P}_i = \frac{\mathrm{P}}{i}$ for $i \in \{r, \ldots n+r-1\}$. This implies that $h_n^{(r)} \notin \mathbb{N}$ if and only if there exists a prime $p \leq n$ such that $\nu_p(h_n^{(r)}) < 0$.

2. Analytic methods

In this section we will prove Theorem 1. For this we need the following corollary of the prime number theorem. For the details of the prime number theory, we refer the reader to [7] or [12].

Fact 5. *For all $\epsilon \in \mathbb{R}_{>0}$, there exists $x_0 = x_0(\epsilon) \in \mathbb{R}$ depending on ϵ such that for all $x \geq x_0$, there exists a prime in the interval $((1-\epsilon)x, x]$.*

Recall that $I(n, r) = \{r, ..., n+r-1\}$. We define $I_p(n, r)$ as the set of integer multiples of p which lie in $I(n, r) = \{r, ..., n+r-1\}$. The following proposition is the first step toward Theorem 1.

Proposition 6. *Let p be a prime number such that $p \leq n$. If $|I_p(n,r)| = 1$, then $h_n^{(r)} \notin \mathbb{N}$. Moreover for a fixed integer r, there exists $n_0 = n_0(r) \in \mathbb{N}$ depending on r such that for all $n \geq n_0$, we have $h_n^{(r)} \notin \mathbb{N}$.*

Proof. First observe that when $n = 2$ the hyperharmonic number $h_n^{(r)} = \frac{2r+1}{2}$ is not an integer. So we may assume that $n \geq 3$. Also we may suppose that $r \geq 2$ as harmonic numbers are not integers except 1. By (1.3), observe that $h_n^{(r)} \notin \mathbb{N}$ if

$$p^{\nu_p(n!)} \nmid \sum_{i=r}^{n+r-1} \mathrm{P}_i. \tag{2.1}$$

Note that $p \nmid \mathrm{P}_i$ if $p \mid i$, and $p \mid \mathrm{P}_i$ if $p \nmid i$ for $i \in I(n,r)$. Since $|I_p(n,r)| = 1$, there is only one such $i \in I(n,r)$ that is divisible by p and hence $p \nmid \sum_{i=r}^{n+r-1} \mathrm{P}_i$. This implies that $h_n^{(r)} \notin \mathbb{N}$ by (2.1). Now we prove the second part of the proposition. Observe that if there exists a prime $p \in \left(\frac{n+r-1}{2}, n\right]$ for $r \geq 2$, then we have $n + r - 1 < 2p$ and so $r - 1 < p$. These inequalities yield that that $|I_p(n,r)| = 1$. By the first part of the proposition, we deduce that $h_n^{(r)} \notin \mathbb{N}$. Therefore it is enough to show that there exists a prime $p \in \left(\frac{n+r-1}{2}, n\right]$. By Fact 5, we know that there exists n_0 such that for all $n \geq n_0$, there is a prime in the interval $(\frac{2n}{3}, n]$; in fact we can take that $n_0 = 2$, since $n \in \mathbb{N}$. Note that when $n \geq 3r - 3$ we have

$$\left(\frac{2n}{3}, n\right] \subseteq \left(\frac{n+r-1}{2}, n\right].$$

Thus we see that given $r \geq 2$ if $n \geq 3r - 3 \geq 2$ then there is always a prime p in the interval $(\frac{n+r-1}{2}, n]$ with $p > \frac{2n}{3} \geq 2r - 2 \geq r$. Hence we conclude that if $r \geq 2$ and $n \geq 3r - 3$ then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. □

Example 7. For instance, one can choose $n = 2p - 1$ and $r = kp + 1$ for some $p \in \mathbb{P}$ and $k \in \mathbb{N}$. In that case $I(n,r) = \{kp + 1, \dots, (k+2)p - 1\}$, and observe that the only multiple of p in $I(n,r)$ is $(k+1)p$. Since $p \in \mathbb{P} \cap \left(p - \frac{1}{2}, 2p - 1\right]$, we conclude that $h_n^{(r)} \notin \mathbb{N}$.

Until now, we have only dealt with the case where $p \in \mathbb{P}$ and $r \leq p < n + r \leq 2p$ as in the previous proposition. However one can notice that it is enough to have $cp \in I(n,r)$, whereas $(c-1)p, (c+1)p \notin I(n,r)$ for some $c \in \mathbb{N}_{\geq 1}$. Therefore one can deduce the following corollary, as a consequence of Proposition 6:

Corollary 8. *For a given $n, r \in \mathbb{N}$, suppose that there exist integers $c, d \geq 1$ and $p, q \in \mathbb{P}$ such that either*

$$(c-1)n \leq r < cn, \qquad \frac{n+r}{c+1} < p < n \quad \textit{or}, \tag{2.2}$$

$$\frac{dn}{2} < r \leq dn, \qquad \frac{n+r}{d+2} < q < \frac{r}{d} \quad \textit{holds}. \tag{2.3}$$

Then in any two cases, we have $h_n^{(r)} \notin \mathbb{N}$.

Proof. For the first part of the theorem, observe that $\frac{n}{2} \leq \frac{n+r}{c+1}$ and $\frac{r}{c} < \frac{n+r}{c+1}$ from (2.2). This implies that $p \in \mathbb{P} \cap \left(\frac{n}{2}, n\right)$ and $r < cp$. The inequality $(c-1)p < (c-1)n \leq r$ can also be shown by using (2.2). Note that $cp < n + r$, because if we have $cp \geq n + r$ then $cp > p + r$ i.e. $(c-1)n > (c-1)p > r$, which is a contradiction. Therefore $(c-1)p < r < cp < n + r < (c+1)p$ holds. So there exists a prime number p such that the conditions $p \in \left(\frac{n}{2}, n\right]$ and $|I_p(n,r)| = 1$ are both satisfied. By Proposition 6, we deduce that $h_n^{(r)} \notin \mathbb{N}$. For the other case, similarly we have $\frac{n}{2} < \frac{n+r}{d+2}$ and $\frac{r}{d} \leq n$. Therefore the prime q lies in $\left(\frac{n}{2}, n\right)$. Moreover since $dq < r$, $n + r < (d+2)q$ and $q < n$, we obtain that $(d+1)q < q + r < n + r$ and $(d+1)q > n + r - q > r$. Again by Proposition 6, we have $h_n^{(r)} \notin \mathbb{N}$. □

As an example of Corollary 8, take $r = 3n$. In that case we can take $c = 4$ or $d = 3$, and then we try to find the lowest n_0 value such that there exists a prime between $\frac{4n}{5}$ and n, for all $n \geq n_0$. We can also use the inequalities given in (2.3) with another value of d, for instance $d = 5$; and try to find a prime in the interval $\left(\frac{4n}{7}, \frac{3n}{5}\right)$. However as one can see that this interval is much restricted than the previous one. So it is better to use smaller values of c and d to obtain more general results.

The following result is the third part of our Theorem 1.

Proposition 9. *For any fixed $n \geq 2$, there are infinitely many values of r such that $h_n^{(r)} \notin \mathbb{N}$. More precisely, denote*

$$\mathcal{R}_n(m) := \left|\left\{r \leq m \,\middle|\, h_n^{(r)} \notin \mathbb{N}\right\}\right|.$$

If $n \geq 2$ is a fixed integer, then $\mathcal{R}_n(m) \gg_n m$ as m tends to infinity. Moreover if $m = m(n)$ is a function of n where $n = o(m)$ then $\mathcal{R}_n(m) \sim m$, as n tends to infinity.

Proof. Let $n \geq 2$ be a fixed integer and $p^{(n)} \in \mathbb{P}$ be the greatest prime that is less than or equal to n, so it is also fixed. By the well-known Bertrand's Postulate, we see that $p^{(n)} \in \left(\frac{n}{2}, n\right]$. Also by Proposition 6, we know that if $|I_{p^{(n)}}(n,r)| = 1$ then $h_n^{(r)} \notin \mathbb{N}$. Define

$$r_c = (c+1) \cdot p^{(n)} - n.$$

Our claim is that if $r \in \bigcup_{c=1}^{\infty} \left((c-1) \cdot p^{(n)}, r_c\right]$ then $h_n^{(r)} \notin \mathbb{N}$. We put $I_c = \left((c-1) \cdot p^{(n)}, r_c\right]$. So suppose that $r \in \mathbb{N}$ is given and there exists $c \in \mathbb{N}$ such that $r \in I_c = \left((c-1) \cdot p^{(n)}, r_c\right]$. In that case,

$$(c-1) \cdot p^{(n)} < r \leq r_c = (c+1) \cdot p^{(n)} - n = c \cdot p^{(n)} - (n - p) \leq c \cdot p^{(n)}$$

holds. Also we have $n + (c-1) \cdot p^{(n)} < n + r \leq (c+1) \cdot p^{(n)}$. By using both of these inequalities, we obtain the condition that $|I_{p^{(n)}}(n,r)| = 1$ as only $cp^{(n)} \in I(n,r)$ and thus $h_n^{(r)} \notin \mathbb{N}$.

Note that $\left|\mathbb{N} \cap ((c-1) \cdot p^{(n)}, r_c]\right| = (c+1) \cdot p^{(n)} - n - (c-1) \cdot p^{(n)} = 2p^{(n)} - n \geq 1$, due to the Bertrand's Postulate. Now let x be a positive real number. Observe that if c and c' are distinct then $I_c \cap I_{c'}$ is empty. Therefore we have

$$\left|\mathbb{N} \cap \bigcup_{c \leq x} \left((c-1) \cdot p^{(n)}, r_c\right]\right| = \sum_{c \leq x} \left|\mathbb{N} \cap \left((c-1) \cdot p^{(n)}, r_c\right]\right| \geq \sum_{c \leq x} 1 \geq x - 1.$$

As a consequence, we deduce that there are infinitely many values of r which satisfies the condition $h_n^{(r)} \notin \mathbb{N}$, for a fixed $n \in \mathbb{N}_{\geq 2}$.

Now let $n \geq 2$ be fixed. Also take $m \in \mathbb{N}$. Define $s = \max\{c \geq 1 \,|\, r_c \leq m\}$. We know by the first part of the proposition that if $r \in \mathbb{N} \cap \bigcup_{c=1}^{\infty} \left((c-1) \cdot p^{(n)}, r_c\right]$ then $h_n^{(r)} \notin \mathbb{N}$. Also by the definition of $\mathcal{R}_n(m)$, we can say that

$$\begin{aligned} \mathcal{R}_n(m) &= \left|\left\{r \leq m \,\middle|\, h_n^{(r)} \notin \mathbb{N}\right\}\right| \\ &\geq \left|\bigcup_{c=1}^{s} \left(\mathbb{N} \cap \left((c-1) \cdot p^{(n)}, r_c\right]\right)\right| \\ &= \sum_{c=1}^{s} (r_c - (c-1) \cdot p^{(n)}). \end{aligned}$$

From the definition of r_c, we obtain that

$$\begin{aligned} \mathcal{R}_n(m) &\geq \sum_{c=1}^{s} ((c+1) \cdot p^{(n)} - n - (c-1) \cdot p^{(n)}) \\ &= \sum_{c=1}^{s} (2p^{(n)} - n) = s \cdot (2p^{(n)} - n). \end{aligned} \tag{2.4}$$

Furthermore for all $c \geq 1$, we have $r_c = (c+1) \cdot p^{(n)} - n \leq (c+1)n - n = cn$. Hence

$$r_s \leq m < r_{s+1} \leq (s+1)n.$$

Therefore we get that

$$s > \frac{m}{n} - 1. \tag{2.5}$$

Combining this fact with (2.4) and Bertrand's postulate, we obtain that

$$\mathcal{R}_n(m) > \frac{m}{n} - 1 \tag{2.6}$$

which leads to the desired inequality. For the last part of the proposition, we will apply [8]. Let n be a sufficiently large fixed integer so that $\mathbb{P} \cap \left(n - n^{0.525}, n\right]$ is not empty. Since

$p^{(n)} \in \left(n - n^{0.525}, n\right]$, we have $2p^{(n)} - n > n - 2n^{0.525}$. Now suppose that $m = m(n)$ is a function of n where $n = o(m)$. Thus (2.4) and (2.5) yield that

$$\mathcal{R}_n(m) > \left(\frac{m}{n} - 1\right) \cdot (n - 2n^{0.525}) = (m - n) \cdot \left(1 - \frac{2}{n^{0.475}}\right). \tag{2.7}$$

From (2.7), we see that $\mathcal{R}_n(m) = m + \mathcal{O}\left(n + \frac{m}{n^{0.475}}\right)$ and this gives the asymptotic $\mathcal{R}_n(m) \sim m$ as n goes to infinity. $\square$

Although the first result given in Proposition 9 can be obtained by the first and the second part of Theorem 2, we need to use the ideas given in the proof of it; so that we can deduce a more general result. In fact, we can extend the result given in Corollary 8 by using Proposition 9 and this is the second part of our Theorem 1.

Theorem 10. *For a given natural number d, there exists a natural number n_d such that if $n \geq n_d$ and $r \leq dn$ then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Moreover there exists a natural number n_0 such that if $n \geq n_0$ and $r \leq \frac{n^{1.475} - 7n}{2}$ then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. In particular, for a fixed r, there are only finitely many possible values of n where $h_n^{(r)}$ may be an integer.*

Proof. By the Prime Number Theorem we know that for a given $d \geq 1$ there exists a real number x_d such that for all $x \geq x_d$ there is a prime in the interval $\left(\left(1 - \frac{1}{2d+7}\right) x, x\right]$. Now we prove the following claim by induction:

Claim. *For a given $d \geq 1$ and for all $n \geq \frac{9}{7} \cdot x_d$, if $r \leq dn$ then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer.*

So first we deal with the case $d = 1$. Let $x_1 = 53$ and $p^{(n)}$ be the greatest prime that is less than or equal to n. Assume that $n \geq x_1$. Then we have that $p^{(n)} \in \left(\frac{8n}{9}, n\right]$. Observe by Proposition 6 and Proposition 9 that if $r \in \mathbb{Z}_{\leq n} \backslash (2p^{(n)} - n, p^{(n)}]$ then $\nu_{p^{(n)}}(h_n^{(r)}) = -1$. So it is enough to check the non-integerness property for $r \in (2p^{(n)} - n, p^{(n)}]$. By (2.3) in Corollary 8, we can say that if $r \in \left(\frac{n}{2}, n\right]$ and there is a prime in the interval $\left(\frac{n+r}{3}, r\right)$ then the corresponding hyperharmonic number is not an integer. Since $r \in (2p^{(n)} - n, p^{(n)}]$ and $p^{(n)} \in \left(\frac{8n}{9}, n\right]$, we get that

$$n \geq p^{(n)} \geq r > 2p^{(n)} - n > \frac{16n}{9} - n = \frac{7n}{9}.$$

In particular we have $r \in \left(\frac{n}{2}, n\right]$. Moreover we obtain that $\frac{n+r}{3} \leq \frac{2n}{3}$. Thus $I_1 := \left(\frac{2n}{3}, \frac{7n}{9}\right] \subseteq \left(\frac{n+r}{3}, r\right)$. Note that if $n \geq \frac{9}{7} \cdot x_1$ then there is a prime in the interval $\left(\frac{2n}{3}, \frac{7n}{9}\right]$. Thus we deduce that for all $n \geq 68 \geq \frac{9}{7} \cdot x_1$ and $r \leq n$, we have $h_n^{(r)} \notin \mathbb{N}$.

Assume that there exists a real number x_{d-1} such that there is a prime in $\left(\left(1 - \frac{1}{2d+5}\right) x, x\right]$ and the claim holds for $d - 1 \geq 1$. Also suppose that for all $x \geq x_d$, the set $\mathbb{P} \cap \left(\left(1 - \frac{1}{2d+7}\right)x, x\right]$ is non-empty. We can assume that $x_d \geq x_{d-1}$. As we did in the case $d = 1$, let $n \geq \frac{9}{7} \cdot x_d$. Then again we have

$$p^{(n)} \in \left(\left(1 - \frac{1}{2d+7}\right)n, n\right]. \tag{2.8}$$

By Proposition 9, we know that the possible values of r that satisfy the property $\nu_{p^{(n)}}(h_n^{(r)}) \geq 0$ belong to the set $\left((j+1)p^{(n)} - n, jp^{(n)}\right]$ where $j \in \{1, \ldots, d\}$. Since $p^{(n)} \in \left(\left(1 - \frac{1}{2d+7}\right)n, n\right]$, we see that for all $j \in \{1, \ldots, d\}$ we have $jp^{(n)} \leq jn < (j+1)p^{(n)}$. This implies that $\left((j+1)p^{(n)} - n, jp^{(n)}\right] \subseteq ((j-1)n, jn]$. So it is enough to check when

$$r \in \left((d+1)p^{(n)} - n, dp^{(n)}\right] \tag{2.9}$$

by the induction hypothesis. We again use (2.3) in Corollary 8 to prove the claim for d. Note that as $r > (d-1)n$ and $d \geq 2$, we see that $r > \frac{dn}{2}$. Also

$$r \leq dp^{(n)} \leq dn \tag{2.10}$$

holds. In order to show the claim, we should see that the interval $\left(\frac{n+r}{d+2}, \frac{r}{d}\right)$ contains a prime number. By (2.8), (2.9) and (2.10) observe that

$$I_d := \left(\frac{d+1}{d+2} \cdot n, \frac{2d^2+6d-1}{2d^2+7d} \cdot n\right] \subseteq \left(\frac{n+r}{d+2}, \frac{r}{d}\right).$$

So finding a prime in the interval I_d leads to the desired result. Note that since

$$\frac{2d^2+6d-1}{2d^2+7d} < \frac{2d+6}{2d+7},$$

the interval I_d does not intersect the interval $\left(\left(1 - \frac{1}{2d+7}\right)n, n\right]$. If we denote $n' = \frac{2d^2+6d-1}{2d^2+7d} \cdot n$, then the lower bound of I_d becomes

$$\frac{d+1}{d+2} \cdot \frac{2d^2+7d}{2d^2+6d-1} \cdot n' = \frac{2d^3+9d^2+7d}{2d^3+10d^2+11d-2} \cdot n'.$$

Computation and simplification indicate that the inequality

$$\frac{2d^3+9d^2+7d}{2d^3+10d^2+11d-2} < \frac{2d+6}{2d+7}$$

holds as we have $5d^2 + 13d - 12 > 0$ for all $d \geq 1$. So if $n' \geq x_d$, then there is a prime in I_d. This can only be achieved when $n \geq \frac{2d^2+7d}{2d^2+6d-1} \cdot x_d$. As

$$n \geq \frac{9}{7} \cdot x_d > \frac{2d^2+7d}{2d^2+6d-1} \cdot x_d,$$

we have $I_d \cap \mathbb{P}$ is non-empty by the fact that the interval $\left(\left(1-\frac{1}{2d+7}\right)x, x\right]$ contains a prime number. By using this fact, we deduce that $h_n^{(r)}$ is not an integer for $n \geq \frac{9}{7} \cdot x_d$ and $r \in ((d-1)n, dn]$. Thus we have the claim. On the other hand, the above proof yields that we can choose $d = d(n)$ as a function of n. We know that by [8], there exists x_0 such that if $n \geq x_0$ then the interval

$$(n - n^{0.525}, n] = \left(n\left(1 - \frac{1}{n^{0.475}}\right), n\right]$$

always contains a prime (actually the number of primes in this interval tends to infinity as n tends to infinity). Thus we can take $2d + 7 = 2d(n) + 7 = n^{0.475}$, in other words $d = d(n) = \frac{n^{0.475}-7}{2}$. Hence if $n \geq n_0 = \left\lceil \frac{9}{7} \cdot x_0 \right\rceil$ and $r \leq dn = \frac{n^{1.475}-7n}{2}$, then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Thus for a fixed r there are only finitely many possible values of n where $h_n^{(r)}$ is an integer. Note that the final result can also be obtained by Proposition 6. □

Some remarks are as follows. As it was mentioned before, we used (2.3) in Corollary 8 to show Theorem 10. Observe that the same result cannot be shown by using (2.2). Moreover the lower bound for n is not the best possible one. For instance for $d = 1$, the lower bound for n can be decreased to 53 instead of 68. One can also take larger interval for $p^{(n)}$. For example if $d = 2$ then we can take $p^{(n)}$ in $\left(\frac{9n}{10}, n\right]$ instead of $\left(\frac{10n}{11}, n\right]$. In that case, we obtain $x_2 = 127$ and there is also a prime in the interval $\left(\frac{3n}{4}, \frac{17n}{20}\right]$ for $n \geq x_2$. However it is necessary to fix the denominator with respect to d to show the theorem by induction. Also the denominator $2d + 7$ is optimal to prove such a theorem with full generality as choosing the denominator of the form $d + k$ does not work where k is an integer.

Finally we prove that almost all hyperharmonic numbers are non-integers which is the first part of Theorem 1 and that also completes the proof of Theorem 1.

Theorem 11. *Let* $S(x) = \left|\left\{(n,r) \in [0,x] \times [0,x] \,\middle|\, h_n^{(r)} \notin \mathbb{Z}\right\}\right|$. *In other words,* $S(x)$ *counts the number of pairs* (n,r) *in the rectangle* $[0,x] \times [0,x]$ *where the corresponding hyperharmonic number* $h_n^{(r)}$ *is not an integer. Then we have* $S(x) = x^2 + \mathcal{O}\left(x^{\frac{2.475}{1.475}}\right)$, *which means that non-integer hyperharmonics have the full asymptotic in the first quadruple. Conditionally, assuming the Riemann hypothesis or Cramér's conjecture, the error term can be taken of size* $\mathcal{O}_\varepsilon\left(x^{\frac{5}{3}+\varepsilon}\right)$ *and* $\mathcal{O}\left(x^{\frac{3}{2}} \cdot \log x\right)$ *respectively.*

Proof. We apply the previous Theorem 10. We know by Theorem 10 if $n \geq n_0$ and

$$r \leq \frac{n^{1.475} - 7n}{2} = \mathcal{O}\left(n^{1.475}\right)$$

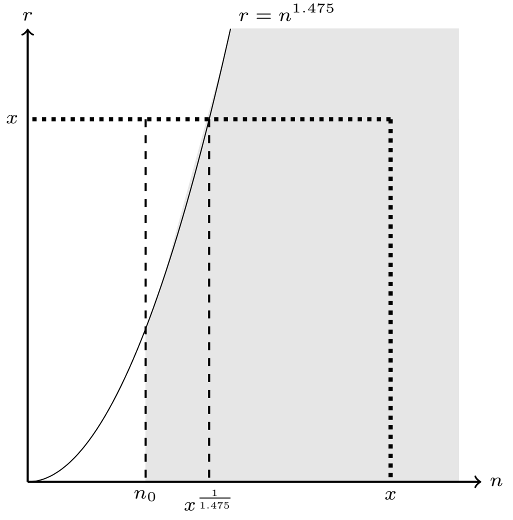

Fig. 1. The graph of $r = n^{1.475}$. The shaded area indicates the lattice points (n, r) where the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer.

then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Now let x be a sufficiently large positive real number. Consider the graph of the function $r = n^{1.475}$ (see Fig. 1).

It intersects the line $r = x$ when $n = x^{\frac{1}{1.475}}$. Therefore we have $S(x) - x^2 \ll n_0 \cdot x + x \cdot x^{\frac{1}{1.475}}$ in other words $S(x) = x^2 + \mathcal{O}\left(x^{\frac{2.475}{1.475}}\right)$ and in particular $S(x) \sim x^2$. Assuming the Riemann hypothesis, in the previous Theorem 10, we can take $d = \mathcal{O}_\varepsilon\left(n^{\frac{1}{2}-\varepsilon}\right)$ and so $dn = \mathcal{O}_\varepsilon\left(n^{\frac{3}{2}-\varepsilon}\right)$. Following the previous idea we have that

$$S(x) - x^2 \ll_\varepsilon n(\varepsilon)x + x \cdot x^{1/(\frac{3}{2}-\varepsilon)} = \mathcal{O}_\varepsilon\left(x^{\frac{5}{3}+\varepsilon}\right).$$

Finally if we assume Cramér's conjecture, we can take $d = \frac{c \cdot n}{\log^2 n}$ for some $c > 0$, and $dn = \frac{c \cdot n^2}{\log^2 n}$. Similarly the graph of the function $r = \frac{c \cdot n^2}{\log^2 n}$ intersects the line $r = x$ when $\frac{\sqrt{c} \cdot n}{\log n} = \sqrt{x}$, and this in turn gives that $n = \mathcal{O}\left(\sqrt{x} \log x\right)$. This yields that $S(x) = x^2 + \mathcal{O}\left(x^{\frac{3}{2}} \cdot \log x\right)$. □

According to the calculations, the power in the error term does not change very much when we use the result given in [8] instead of the Riemann Hypothesis. In fact, the difference between the powers is approximately 0.0113.

3. Combinatorial methods

In this section we will prove Theorem 2. Before beginning its proof recall that we define the set $I_a(n, r) := a\mathbb{Z} \cap [r, n + r)$ in the previous section, where a is a given prime number. In this section we use the same notation for an arbitrary positive integer a, instead of just a prime. And as a start for this section, we give a relationship between the number of elements of $I_a(n,r)$ and $I_a(n,1)$, for any $a \in \mathbb{N}_{>0}$.

Lemma 12. *Let a be a given positive integer and define $k = |I_a(n,1)| = \left\lfloor \frac{n}{a} \right\rfloor$. Then we have*

$$|I_a(n,r)| \in \{k, k+1\}\,.$$

Moreover if $a \mid n$, then $|I_a(n,r)| = k$.

Proof. Let $b = |I_a(n,r)|$ and $c = \left\lceil \frac{r}{a} \right\rceil$. Since $n < (k+1)a$ and $r \le ca$, we have $n + r < (k+1+c)a$; or in other words

$$(c+b-1)a \le n+r-1 < (c+k+1)a.$$

Hence $b-1 < k+1$ which is equivalent to $b < k+2$. Also since $ka \le n$ and $(c-1)a < r$, we have $(c+k-1)a < n+r$. Therefore,

$$(c+k-1)a \le n+r-1 < (c+b)a$$

which leads to the fact that $k-1 < b$. Thus, $b \in \{k, k+1\}$.

Moreover, if a divides n then $n = ka$. Hence $(n+r-1) = ka + r - 1 \le (k+c)a - 1$; and this is equivalent to $(b-k-1)a \le -1$. So b cannot be equal to $k+1$. Thus, $|I_a(n,r)| = b = k = \frac{n}{a}$. $\square$

Definition 13 *(Maximal power of a prime in a set).* Let I be a finite set and p be a prime. We define $\mu_p(I) = \max\{\nu_p(s)\,|\, s \in I\}$ as *the maximal power of p in I*. The number of integers in $\left(p^{\mu_p(I)}\mathbb{Z}\right) \cap I$ is denoted by $\mathcal{M}_p(I)$.

Note that for any non-empty set I and $p \in \mathbb{P}$, we have $\mathcal{M}_p(I) \ge 1$. Also for any $I = I(n,r)$, observe that $\mathcal{M}_p(I) < p$: if $\mathcal{M}_p(I) \ge p$ then there exists $c \in \mathbb{N}_{\ge 1}$ such that $cp^\theta, (c+1)p^\theta, \dots (c+p-1)p^\theta \in I$, where θ denotes the $\mu_p(I)$. By the definition of θ, we deduce that $p \nmid c+i$ for all $i \in \{0, \dots, p-1\}$. However p must divide either one of the p consecutive integers. This leads to a contradiction.

Example 14. Take $r = 1$ and let $p \in \mathbb{P}$ and $n \in \mathbb{Z}_{\ge 2}$ be given where $p^\alpha \le n < p^{\alpha+1}$ holds for some $\alpha \in \mathbb{N}$. In that case, we will have

$$\mu_p(I(n,1)) = \alpha = \lfloor \log_p n \rfloor$$

where $I(n,r) = I(n,1) = \{1, 2, \dots, n\}$.

Lemma 15. *Let $n \geq 2$ and $r \geq 1$ be given. Then for all primes $p \leq n$, we have*

$$\lfloor \log_p n \rfloor \leq \mu_p(I(n,r)) \leq \lfloor \log_p(n+r-1) \rfloor.$$

Proof. Let $\theta = \mu_p(I(n,r))$. By definition, there exists $a \in I(n,r)$ such that $a = cp^\theta$ for some $c \in \mathbb{Z}^+$ where $p \nmid c$. Therefore, $p^\theta \leq a = cp^\theta \leq n+r-1$ which yields that $\theta \leq \log_p(n+r-1)$. As θ is in $\mathbb{N}$, we have $\theta \leq \lfloor \log_p(n+r-1) \rfloor$.

Let $p^\alpha \leq n < p^{\alpha+1}$ be given. So to show the second inequality, we assume that $\mu_p(I(n,r)) = \theta < \alpha$. That means by definition, for all $c \in \mathbb{N}$, $cp^\alpha \notin I(n,r)$. In other words, there exists $c_0 \in \mathbb{N}$ such that $c_0 p^\alpha < r$ and $n+r-1 < (c_0+1)p^\alpha$. This leads to $n < p^\alpha$, which is a contradiction. □

These bounds are sharp indeed. For example, if we take $n = 4$, $r = 25$ and $p = 2$, then $I(n,r) = \{25, 26, 27, 28\}$ and therefore $\mu_p(I(n,r)) = 2 = \lfloor \log_p n \rfloor$.

In order to prove Theorem 2, we need to use some combinatorial tools. In particular, we use the following lemma which can be found in either [17, p. 116] or [15]:

Lemma 16 *(Kummer). Let a, b be given two natural numbers. Then $\nu_p\left(\binom{a+b}{b}\right)$ is equal to the number of carries that occur in the addition of a and b in their p-ary representations.*

The following proposition will be a key step for Theorem 2.

Proposition 17. *Let p be a given prime number. Assume that $n = (n_\alpha, n_{\alpha-1}, \ldots, n_0)_p \in \mathbb{N}_{\geq 2}$ and $r-1 = (r'_\beta, r'_{\beta-1}, \ldots, r'_0)_p \in \mathbb{N}$ are the p-ary representations of n and $r-1$, respectively. Also let $\delta = \max\{\alpha, \beta\}$ and $n+r-1 = \sum_{i=0}^{\delta+1} s_i p^i = (s_{\delta+1}, s_\delta, \ldots, s_0)_p$ where $s_{\delta+1} \in \{0,1\}$. Then*

$$\mu_p(I(n,r)) = \max\{i \in \{0,1,\ldots,\delta+1\} \,|\, s_i > r'_i\}. \tag{3.1}$$

Moreover we have

$$\nu_p\left(\binom{n+r-1}{r-1}\right) \leq \mu_p(I(n,r)).$$

Proof. Let $\gamma := \max\{i \in \{0,1,\ldots,\delta+1\} \,|\, s_i > r'_i\}$ and $\theta := \mu_p(I(n,r))$. We want to prove that $\theta = \gamma$. Note that this γ exists because if not, then for all $i \in \{0,1,\ldots,\delta+1\}$ the inequality $s_i \leq r'_i$ holds. This implies that $n+r-1 = \sum_{i=0}^{\delta+1} s_i p^i \leq \sum_{i=0}^{\delta} r'_i p^i = r-1$, which is a contradiction since $n > 0$. Furthermore we claim that for all $j > \gamma$ we have $s_j = r'_j$. To show this suppose contrary, i.e. take the largest $j > \gamma$ such that $s_j < r'_j$, and call it j_0. Then consider

$$0 > r-1-(n+r-1) = \sum_{i=0}^{\delta} r_i' p^i - \sum_{i=0}^{\delta+1} s_i p^i = \sum_{i=0}^{j_0} (r_i' - s_i) p^i$$
$$= (r_{j_0}' - s_{j_0}) p^{j_0} + \sum_{i=0}^{j_0-1} (r_i' - s_i) p^i \geq p^{j_0} - \sum_{i=0}^{j_0-1} (p-1) p^i$$
$$= p^{j_0} - (p-1) \cdot \frac{p^{j_0}-1}{p-1} \geq 1.$$

A contradiction, hence we have the claim. Now consider

$$b = \sum_{i=\gamma}^{\delta+1} s_i p^i$$

where $s_{\delta+1}$ is possibly zero. We first have to show that $b \in I(n,r)$. To do this, we represent $r-1$ as $\sum_{i=0}^{\delta+1} r_i' p^i$ where $r_{\beta+1}' = \ldots = r_{\delta+1}' = 0$. It is easy to see that

$$b = \sum_{i=\gamma}^{\delta+1} s_i p^i \leq \sum_{i=0}^{\delta+1} s_i p^i = n+r-1.$$

Also by definition of b and γ, we have $s_i = r_i'$ for all $i \in \{\gamma+1, \ldots, \delta+1\}$ and $s_\gamma > r_\gamma'$; which implies that $b > r-1$. If $\{\gamma+1, \ldots, \delta+1\}$ is empty, then $\gamma = \delta+1$ and by definition of δ we have again $b > r-1$; as $s_{\delta+1} > r_{\delta+1}' = 0$. Thus $b \in I(n,r)$; and hence one can easily see that $\gamma = \nu_p(b) \leq \mu_p(I(n,r)) = \theta$. So it is enough to prove that $\theta \leq \gamma$.

If $\gamma = \delta+1$, then $\gamma = \lfloor \log_p(n+r-1) \rfloor$ and hence by Lemma 15 we have $\theta \leq \gamma$. For the second case, suppose that $\gamma < \delta+1$. That means $s_{\delta+1} = r_{\delta+1}'$ and since $\beta \leq \delta$, we see that $s_{\delta+1} = 0$. Also assume that $\theta > \gamma$. Since $\theta, \gamma \in \mathbb{Z}$, we have $\theta \geq \gamma+1$. By definition of $\mu_p(I(n,r))$, there exists $a \in I(n,r)$ such that $a = c_a p^\theta$ for some $c_a \in \mathbb{N}$ and $p \nmid c_a$. In other words, we have $a = \sum_{i=0}^{\delta} a_i p^i \in I(n,r)$ where $a_0 = a_1 = \ldots = a_{\theta-1} = 0$ and $a_\theta \neq 0$.

If $\delta = \alpha$, then since $n_\alpha > 0$ we have $s_\delta > r_\delta$. This implies that $\gamma \geq \delta$, but since $\gamma < \delta+1$, this yields that $\gamma = \delta$. However by Lemma 15 we have $\theta \leq \delta = \lfloor \log_p(n+r-1) \rfloor$, which leads to a contradiction. Thus $\delta > \alpha$ as $\delta = \max\{\alpha, \beta\}$.

So assume that $\delta = \beta$, or more precisely $\beta > \alpha$. Due to the definition of γ, we have $s_i = r_i'$ for all $i \in \{\gamma+1, \ldots, \delta\}$. Also since $a \in I(n,r)$, observe that $a = \sum_{i=\theta}^{\delta} a_i p^i \leq \sum_{i=\theta}^{\delta} s_i p^i$. Because otherwise the inequality $a - \sum_{i=\theta}^{\delta} s_i p^i = c_0 p^\theta > 0$ holds for some $c_0 \in \mathbb{N}_{\geq 1}$. Therefore

$$\sum_{i=0}^{\theta-1} s_i p^i \leq \sum_{i=0}^{\theta-1} (p-1) p^i = (p-1) \cdot \frac{p^\theta - 1}{p-1} < p^\theta \leq c_0 p^\theta = a - \sum_{i=\theta}^{\delta} s_i p^i.$$

This implies that $a > \sum_{i=0}^{\delta} s_i p^i = n+r-1$, which is impossible. So by using this fact and the inequality $\theta \geq \gamma+1$, we obtain that

$$a = \sum_{i=\theta}^{\delta} a_i p^i \leq \sum_{i=\theta}^{\delta} s_i p^i \leq \sum_{i=\gamma+1}^{\delta} s_i p^i = \sum_{i=\gamma+1}^{\delta} r_i' p^i \leq \sum_{i=0}^{\delta} r_i' p^i = r-1$$

which is a contradiction. Thus $\theta \leq \gamma$ and hence we have the equality.

To prove the second part, we use Lemma 16 and apply the first part of the proposition. Define $n_i = 0$, for all $i \in \{\alpha+1, \ldots, \delta+1\}$. Observe that there can't be any carry for the values of $i = \theta+1, \ldots, \delta+1$, due to (3.1). The situation is also same for the case $i = \theta$. Because if a carry occurs for that case, then $s_{\theta+1} \geq r'_{\theta+1} + 1$ which contradicts the fact that $s_i = r_i'$ for all $i \in \{\theta+1, \ldots, \delta+1\}$. This in turn implies that the only possible carries in the addition of n and $r-1$ in their p-ary representations are for the entries $i = 0, \ldots, \theta-1$. Thus, there are at most θ carries in the addition; and hence $\nu_p\left(\binom{n+r-1}{r-1}\right) \leq \theta = \mu_p(I(n,r))$. □

Due to Lemma 16 and Proposition 17, we observe that it is enough to find digits where a carry does not occur in the addition of n and $r-1$ in their p-ary representations in order to obtain $\mu_p(I(n,r)) > \nu_p\left(\binom{n+r-1}{r-1}\right)$. This fact will also help us to find pairs (n,r) where the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. The following theorem is a part of Theorem 2:

Theorem 18. *Let n, r be given natural numbers. Suppose that there exists a prime number p such that $\mu_p(I(n,r)) > \nu_p\left(\binom{n+r-1}{r-1}\right)$ and $\mathcal{M}_p(I(n,r)) = 1$. Then $h_n^{(r)} \notin \mathbb{N}$. In particular for all $n \geq 2$ and $r \geq 1$, if n is even or r is odd then $h_n^{(r)} \notin \mathbb{N}$. Moreover for all $\alpha \geq 1$ and prime p, we have $h_{p^\alpha}^{(r)} \notin \mathbb{N}$.*

Proof. Let $\theta = \mu_p(I(n,r))$. From (1.2), we can see that

$$\begin{aligned} h_n^{(r)} \in \mathbb{N} &\Leftrightarrow \forall p \in \mathbb{P} \quad \nu_p\left(h_n^{(r)}\right) \geq 0 \\ &\Leftrightarrow \forall p \in \mathbb{P} \quad \nu_p\left(\binom{n+r-1}{r-1}\right) + \nu_p\left(h_{n+r-1} - h_{r-1}\right) \geq 0. \end{aligned} \tag{3.2}$$

So take the prime number p which satisfies the given assumptions of the theorem. Then consider

$$h_{n+r-1} - h_{r-1} = \sum_{i=r}^{n+r-1} \frac{1}{i} = \frac{1}{cp^\theta} + \sum_{\substack{i=r \\ i \neq cp^\theta}}^{n+r-1} \frac{1}{i}, \tag{3.3}$$

where $p \nmid c$. Since $\mathcal{M}_p(I(n,r)) = 1$, we can say that the maximal power of p in the set $I(n,r) \setminus \left\{cp^\theta\right\}$ is at most $\theta - 1$. By the non-Archimedean property of p-adic valuation, we have

$$\nu_p\left(\sum_{\substack{i=r\\ i\neq cp^\theta}}^{n+r-1}\frac{1}{i}\right) > -\theta. \tag{3.4}$$

Again by the same property, one can also see that $\nu_p(h_{n+r-1}-h_{r-1}) = -\theta$. Since $\theta > \nu_p\left(\binom{n+r-1}{r-1}\right)$, we obtain that $\nu_p(h_n^{(r)}) < 0$. By (3.2), we conclude that $h_n^{(r)} \notin \mathbb{N}$.

For the second part of the theorem assume that $2^{\alpha_2} \leq n < 2^{\alpha_2+1}$ holds, where $\alpha_2 \geq 1$. Then we have $\frac{n}{2} < 2^{\alpha_2} \leq n$. This implies that for all $r \in \mathbb{N}$ we have $|I_{2^{\alpha_2}}(n,r)| \in \{1,2\}$, due to Lemma 12. Define $\theta_2 = \mu_2(I(n,r))$. Since $\theta_2 \geq \alpha_2$, we have $I_{2^{\theta_2}}(n,r) \subseteq I_{2^{\alpha_2}}(n,r)$. Therefore $\mathcal{M}_2(I(n,r)) = |I_{2^{\theta_2}}(n,r)| \in \{1,2\}$. We will show that $|I_{2^{\theta_2}}(n,r)| = 1$. Suppose not. Then by Definition 13 there exists $c \in \mathbb{N}$ such that $c2^{\theta_2}, (c+1)2^{\theta_2} \in I(n,r)$. However either c or $c+1$ is even and this implies that $\mu_2(I(n,r)) \geq \theta_2 + 1$, which contradicts the definition of θ_2. Thus $\mathcal{M}_2(I(n,r)) = 1$.

Without loss of generality, suppose that n is even. If we write $n = (n_\alpha, \ldots, n_0)_2$, then we have $n_0 = 0$. This implies that there doesn't exist any carry in the first step of the addition of n and $r-1$. Due to Lemma 16 and Proposition 17, we obtain that $\nu_2\left(\binom{n+r-1}{r-1}\right) < \theta_2$; and thus $h_n^{(r)} \notin \mathbb{N}$. The case that r is odd follows similarly.

For the last part, we will follow the same lines that we did for the case $p = 2$. Let $\theta_p := \mu_p(I(n,r))$. Since $n = p^\alpha$, observe that $\mathcal{M}_p(I(n,r)) = \left|I_{p^{\theta_p}}(n,r)\right| = 1$. Also if $n = (n_\alpha, n_{\alpha-1}, \ldots, n_0)_p$, we have $n_\alpha = 1$ and $n_i = 0$, for all $i \in \{0, \ldots, \alpha-1\}$. This implies that there exist $\alpha \geq 1$ many entries such that a carry does not occur in the addition of n and $r-1$. Thus we have

$$\nu_p\left(\binom{n+r-1}{r-1}\right) \leq \theta_p - \alpha \leq \theta_p - 1 < \theta_p.$$

And hence $h_n^{(r)} \notin \mathbb{N}$. $\square$

Remark 19. Note that for the case $p = 2$, it is enough to obtain a digit where a carry does not occur in the binary addition of n and $r-1$. Let $n = (n_\alpha, n_{\alpha-1}, \ldots, n_0)_2 \in \mathbb{N}_{\geq 2}$ and $r-1 = (r'_\beta, r'_{\beta-1}, \ldots, r'_0)_2 \in \mathbb{N}$ be binary representations of n and $r-1$. Define $\delta = \max\{\alpha, \beta\}$ and $n_i = r'_j = 0$, for all $i \in \{\alpha+1, \ldots, \delta\}$ and $j \in \{\beta+1, \ldots, \delta\}$. Observe that if there exists $t \leq \delta$ such that $n_t = r'_t = 0$ then $\nu_2\left(\binom{n+r-1}{r-1}\right) < \theta_2$ even if a carry occurs in the addition of the previous values. In that case for $n \geq 2$, the hyperharmonic number $h_n^{(r)}$ is not an integer by Theorem 18.

The case that we mentioned in the previous paragraph can be satisfied when both n_t and r'_t are even simultaneously for some $t \in \{0, \ldots, \delta\}$. In fact by Theorem 18, we know that n and $r-1$ should be odd if $h_n^{(r)}$ is an integer. So even this is the case, namely $n_0 = r'_0 = 1$, then $n_1 = r'_1 = 0$ implies that $h_n^{(r)} \notin \mathbb{N}$. This situation holds when both $n \equiv r-1 \equiv 1 \bmod 4$. Consequently if $n \equiv 1 \bmod 4$ and $r \equiv 2 \bmod 4$, we conclude that

$h_n^{(r)} \notin \mathbb{N}$. Note that this process can be developed to obtain similar equivalence classes in any power of 2.

Remark 20. When we consider the primes other than 2, we may not obtain the same result. For instance, take $p = 3$ and $n \geq 3$. Then for any $r \in \mathbb{N}$ we have $\mathcal{M}_3(I(n,r)) \in \{1,2\}$. By Theorem 18 if $\mathcal{M}_3(I(n,r)) = 1$ and $\nu_3\left(\binom{n+r-1}{r-1}\right) < \mu_3(I(n,r))$, then we obtain that $h_n^{(r)} \notin \mathbb{N}$. However if $\mathcal{M}_3(I(n,r)) = 2$, we might not get the condition $\nu_3(h_n^{(r)}) < 0$ even if $\nu_3\left(\binom{n+r-1}{r-1}\right) < \mu_3(I(n,r))$. Because in that case, there exists $c \in \mathbb{N}_{\geq 1}$ such that $c \equiv 1 \bmod 3$ and $c3^{\theta_3}, (c+1)3^{\theta_3} \in I(n,r)$, where $\theta_3 = \mu_3(I(n,r))$. Note that c cannot be equivalent to $0, 2 \bmod 3$; because if so then either one of c or $c+1$ is a multiple of 3, and hence $\mathcal{M}_3(I(n,r)) = 1$. This implies that

$$h_{n+r-1} - h_{r-1} = \frac{1}{c3^{\theta_3}} + \frac{1}{(c+1)3^{\theta_3}} + \sum_{\substack{i=r \\ 3^{\theta_3} \nmid i}}^{n+r-1} \frac{1}{i} = \frac{2c+1}{c(c+1)3^{\theta_3}} + q,$$

where q denotes the sum of reciprocals of the elements of $I(n,r)$ which are not divisible by 3^{θ_3}. Similar to (3.4) we can say that $\nu_3(q) \geq 1 - \theta_3$, due to the non-Archimedian property of the p-adic valuation. As $c \equiv 1 \bmod 3$, we see that $2c+1 \equiv 0 \bmod 3$ and $\nu_3(c) = \nu_3(c+1) = 0$. Thus we have $\nu_3\left(\frac{2c+1}{c(c+1)3^{\theta_3}}\right) = \nu_3(2c+1) - \theta_3 \geq 1 - \theta_3$; and hence

$$\nu_3(h_{n+r-1} - h_{r-1}) \geq \min\left\{\nu_3\left(\frac{2c+1}{c(c+1)3^{\theta_3}}\right), \nu_3(q)\right\} \geq 1 - \theta_3.$$

So if $\nu_3\left(\binom{n+r-1}{r-1}\right) = \theta_3 - 1 < \mu_3(I(n,r))$, then by (1.2) we get

$$\nu_3(h_n^{(r)}) = \nu_3\left(\binom{n+r-1}{r-1}\right) + \nu_3(h_{n+r-1} - h_{r-1}) \geq 0.$$

Consequently for some specific $r \in \mathbb{N}$, the inequality $\nu_3(h_n^{(r)}) \geq 0$ can be obtained.

Nevertheless Theorem 18 can be used to cover most of the (n,r)-tuples where $h_n^{(r)} \notin \mathbb{N}$. More precisely, one can give some specific values of n and r which may depend on a prime p such that $\mu_p(I(n,r)) > \nu_p\left(\binom{n+r-1}{r-1}\right)$ and $\mathcal{M}_p(I(n,r)) = 1$. However, we do not mention here each case separately. Instead we will give the following theorem which may widely be used to cover most of the cases and it is the other part of Theorem 2:

Theorem 21. *Let a be nonnegative and m, k, b be positive integers. Let p be a prime number such that $a + b \leq p$. Put*

$$n = mp + a = (n_\alpha, \dots, n_0)_p$$

$$r = kp + b = (r_\beta, \dots, r_0)_p,$$

where $(n_\alpha, \dots, n_0)_p$ and $(r_\beta, \dots, r_0)_p$ are the p-ary representations of n and r, respectively. If there exist $c \in \mathbb{N}$ and $\kappa > \alpha$ such that $r \in (cp^\kappa - n, cp^\kappa]$, then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Otherwise the non-integerness property still holds, if p is sufficiently large depending on m and k.

Proof. Let $\theta = \mu_p(I(n,r))$. We will show that the existence of $c \in \mathbb{N}$ and $\kappa > \alpha$ where $cp^\kappa - n < r \leq cp^\kappa$ is equivalent to $\theta > \alpha$. Note that this condition implies that $r \leq cp^\kappa < n + r$. Hence $\theta \geq \nu_p(c) + \kappa > \alpha$. For the vice versa, if $\theta > \alpha$ then there exists $c_0 \in \mathbb{N}$ such that $r \leq c_0 p^\theta < n + r$. This indicates that $r \in (c_0 p^\theta - n, c_0 p^\theta]$. Taking $\kappa = \theta$ and $c = c_0$ lead to the condition on r given in the theorem.

Observe that $n_0 = a$ and $r_0 = b > 0$. Since $0 < a + b \leq p$, a carry does not occur in the first step of the addition of $n = (n_\alpha, \dots, n_1, a)_p$ and $r - 1 = (r_\beta, \dots, r_1, b-1)_p$. This implies that $\nu_p\left(\binom{n+r-1}{r-1}\right) < \theta$.

So for the first part of the theorem, assume that $\theta > \alpha$. Then we see that $n < p^\theta$, since

$$n = \sum_{i=0}^{\alpha} n_i p^i \leq \sum_{i=0}^{\alpha} (p-1)p^i = p^{\alpha+1} - 1.$$

By Lemma 12, we obtain that

$$\mathcal{M}_p(I(n,r)) = \left|I_{p^\theta}(n,r)\right| \in \left\{ \left\lfloor \frac{n}{p^\theta} \right\rfloor, \left\lfloor \frac{n}{p^\theta} \right\rfloor + 1 \right\} = \{0, 1\}.$$

Since $I(n,r)$ is non-empty, we deduce that $\mathcal{M}_p(I(n,r)) = 1$. By the inequality $\nu_p\left(\binom{n+r-1}{r-1}\right) < \theta$ and Theorem 18, we deduce that $h_n^{(r)} \notin \mathbb{N}$.

For the second part of the theorem, we may assume that $\theta = \alpha$ by the first part of the theorem and by Lemma 15. Define $d = \left\lceil \frac{r}{p^\alpha} \right\rceil$ and

$$s = \mathcal{M}_p(I(n,r)) = |I_{p^\alpha}(n,r)|.$$

Note that d and s depend on k and m, respectively. Consider

$$h_{n+r-1} - h_{r-1} = \frac{1}{p^\alpha} \cdot \left(\frac{1}{d} + \dots + \frac{1}{d+s-1} \right) + \sum_{\substack{i=r \\ p^\alpha \nmid i}}^{n+r-1} \frac{1}{i}. \tag{3.5}$$

In order to obtain the p-adic valuation of $h_{n+r-1} - h_{r-1}$, we have to calculate the p-adic valuation of each term on the right hand side. So first of all, consider the second summand. Observe that the largest power of p that can divide the denominator of one of these reciprocals in this sum is $\alpha - 1$. By the non-Archimedian property of the p-adic valuation, we find that

$$\nu_p\left(\sum_{\substack{i=r\\ p^\alpha \nmid i}}^{n+r-1} \frac{1}{i}\right) \geq 1-\alpha. \tag{3.6}$$

For the first summand since

$$\left\lfloor \frac{n}{p^\alpha} \right\rfloor = \left\lfloor n_\alpha + \sum_{i=0}^{\alpha-1} n_i p^{i-\alpha} \right\rfloor = n_\alpha \quad \text{and}$$

$$\left\lceil \frac{r}{p^\alpha} \right\rceil = \left\lceil \sum_{i=\alpha}^{\beta} r_i p^{i-\alpha} + \sum_{i=0}^{\alpha-1} r_i p^{i-\alpha} \right\rceil = \sum_{i=0}^{\beta-\alpha} r_{\alpha+i} p^i + 1,$$

we get $d = (r_\alpha + 1) + \sum_{i=1}^{\beta-\alpha} r_{\alpha+i} p^i$ and $s \in \{n_\alpha, n_\alpha + 1\}$ by Lemma 12. Note that none of $j \in \{d, \ldots, d+s-1\}$ is a multiple of p. Because if not then there is $\ell \in \mathbb{N}$ such that $j = \ell p$ and $jp^\alpha = \ell p^{\alpha+1} \in I(n,r)$. In that case $\mu_p(I(n,r)) = \alpha+1 > \theta$, which contradicts the definition of θ.

We set

$$z = \sum_{j=d}^{d+s-1} \frac{1}{j} = \frac{\mathrm{N}_z}{\mathrm{D}_z},$$

where N_z and D_z denote the numerator and the denominator of z, respectively. We also assume that $\gcd(\mathrm{N}_z, \mathrm{D}_z) = 1$. If p is greater than $\mathrm{N}_z > 0$, then $\nu_p(\mathrm{N}_z) = 0$. Moreover since

$$\{d, \ldots, d+s-1\} \cap p\mathbb{Z} = \emptyset,$$

we see that $p \nmid d \cdot (d+1) \cdots (d+s-1)$. Since $\mathrm{D}_z \mid d \cdot (d+1) \cdots (d+s-1)$ we deduce that $p \nmid \mathrm{D}_z$, and so $\nu_p(\mathrm{D}_z) = 0$. These facts imply that $\nu_p(z) = 0$ and hence

$$\nu_p\left(\frac{1}{p^\alpha} \cdot \left(\frac{1}{d} + \cdots + \frac{1}{d+s-1}\right)\right) = -\alpha + \nu_p\left(\frac{\mathrm{N}_z}{\mathrm{D}_z}\right) = -\alpha.$$

Combining this fact with (3.6) leads to $\nu_p(h_{n+r-1} - h_{r-1}) = -\alpha$, due to the non-Archimedian property again. Recall that $\nu_p\left(\binom{n+r-1}{r-1}\right) < \theta = \alpha$. Thus we conclude by (3.2) that $h_n^{(r)} \notin \mathbb{N}$. $\square$

This theorem simply eliminates most of the cases. For instance let $p \in \mathbb{P}$, $n = mp + a$ for some $a \in \{0, \dots, p-1\}$, $\alpha \geq 1$ and $m \in \{1, \dots, p^{\alpha} - 1\}$. If $\kappa > \alpha$ and $r = kp^{\kappa} - i$ for some $k \in \mathbb{N}$ and $i \in \{a, a+1, \dots, mp-1\} \cap \{i = (i_{\alpha}, \dots, i_0)_p \mid a \leq i_0 < p\}$, then we can see that $h_n^{(r)} \notin \mathbb{N}$ due to the fact that $kp^{\kappa} \in I(n, r)$. Therefore one can choose appropriately the values of m, p, a, k, κ and i to show that $h_n^{(r)}$ is not an integer for a huge class of pairs (n, r). Moreover for $r = kp + b$ as in Theorem 21, one can fix m, b, a and k to check the non-integerness property of $h_{mp+a}^{(kp+b)}$ for different values of p. However we do not discuss each case separately here. Nevertheless we would like to point out that these classes of (n, r) tuples cannot be obtained from analytic methods, especially when r is large.

Example 22. To give an example for Theorem 21, we first take $n = 33$ and $p = 31$. In that case we have $a = 2$ and $m = 1$. Also take $r = 11.082.234 = 357491 \cdot 31 + 13$. Note that $r \in (12 \cdot 31^4 - 33, 12 \cdot 31^4]$. This implies that $h_n^{(r)}$ is not an integer for the given values of n and r. Observe that this result may not be achieved by using analytic methods. Namely if we use Theorem 10, the only way to achieve this $12 \cdot 31^4$ bound may be to set $d = \left\lceil \frac{12 \cdot 31^4}{33} \right\rceil = 335.826$ and try to find the corresponding n_d so that for all $n \geq n_d$, $r \leq dn$ implies that $h_n^{(r)} \notin \mathbb{N}$. However the corresponding n_d bound may be far beyond the value of $n = 33$, because we know by previous examples that the corresponding lower bound for $d = 1$ is 53.

For the second part of the theorem, we will calculate the corresponding upper bound for r where $n = (3, 0, \dots, 0)_p = 3p^{\alpha}$ and $\alpha \geq 1$. Since $n = 3p^{\alpha}$ and $s = |p^{\alpha}\mathbb{Z} \cap I(n, r)|$ we know that $s = 3$. Moreover n is of the form $mp + a$ where $m = 3p^{\alpha-1}$ and $a = 0$. Therefore we can take $r = kp + b$ where $b \in \{1, \dots, p-1\}$. Note that we can also take $b = 0$ since a carry does not occur in the first step of the addition of n and $r - 1$ in their p-ary representations due to the fact that $a = 0$. So we do not have any restrictions on r, except for a possible upper bound depending on p. If the numerator of

$$\frac{1}{d} + \frac{1}{d+1} + \frac{1}{d+2} = \frac{3d^2 + 6d + 2}{d(d+1)(d+2)}$$

is less than p where $d = \left\lceil \frac{r}{p^{\alpha}} \right\rceil$, then we deduce by the second part of Theorem 21 that $h_n^{(r)} \notin \mathbb{N}$. Hence it is enough to check the primes where $p > 3d^2 + 6d + 2$. Observe that $\frac{3r^2}{p^{2\alpha}} + \frac{12r}{p^{\alpha}} + (11 - p) < 0$ implies $p > 3d^2 + 6d + 2$. This inequality is satisfied when the inequalities $-2 - p^{\alpha}\sqrt{\frac{p+1}{3}} < r < -2 + p^{\alpha}\sqrt{\frac{p+1}{3}}$ hold. Thus $r < p^{\alpha}\sqrt{\frac{p+1}{3}} - 2$ leads to the fact that $h_n^{(r)}$ is not an integer.

Now we give a more general result similar to the previous example where we can compare the power of combinatorial and analytic methods on non-integerness problem of hyperharmonic numbers. Let $n = sp^{\alpha}$ for some $\alpha, s \in \mathbb{N}_{\geq 1}$ and a prime p. For $s = 1$ and even values of s, we know by Theorem 18 that $h_n^{(r)}$ is not an integer for all values of r. Also if $s = 3$ then we can take $r = \mathcal{O}\left(n \cdot n^{\frac{1}{2\alpha}}\right)$ due to the Example 22. For $\alpha = 1$, the upper bound for r can be given as $\mathcal{O}\left(n\sqrt{n}\right)$. A slightly weaker result can be obtained from analytic methods under the Riemann Hypothesis, i.e. r can be taken as $\mathcal{O}\left(n^{\frac{3}{2}-\epsilon}\right)$ for all $\epsilon > 0$. Therefore it is better to use combinatorial methods instead of analytic ones when $s \leq 3$ and $\alpha = 1$.

However, when we take an odd $s > 3$, we won't get any better results compared to the analytic case. To explain this fact more precisely, first recall that

$$z = \frac{s}{\mathrm{H}} = \frac{\frac{s\mathrm{P}}{\mathrm{H}}}{d(d+1)\cdots(d+s-1)},$$

where P denotes the product and H denotes the harmonic mean of $d, d+1, \ldots, d+s-1$. Note that $\frac{s\mathrm{P}}{\mathrm{H}}$ is an integer and $d = \left\lceil \frac{r}{p^\alpha} \right\rceil$ as in Example 22. By the well-known bounds on H, we can say that the numerator of z is equal to

$$\frac{s\mathrm{P}}{\mathrm{H}} \leq \frac{s\mathrm{P}}{d} = s(d+1)\cdots(d+s-1).$$

So if $p > s(d+1)\cdots(d+s-1)$ we obtain that $h_n^{(r)} \notin \mathbb{N}$. This implies that d can be taken of size $\mathcal{O}\left(p^{\frac{1}{s-1}}\right)$, for a fixed s. Since $d = \left\lceil \frac{r}{p^\alpha} \right\rceil$, we deduce that

$$r = \mathcal{O}\left(p^{\alpha + \frac{1}{s-1}}\right) = \mathcal{O}\left(n^{1+\frac{1}{\alpha(s-1)}}\right).$$

Unfortunately this bound does not lead to better results compared to the analytic case, even if we take $\alpha = 1$. Note that the order of the magnitude can be obtained from the analytic approach is 1.475, due to Theorem 10; whereas the corresponding bound here is at most $1 + \frac{1}{\alpha(s-1)} \leq 1.25$. Therefore it is necessary to use different methods in order to obtain better bounds for r.

4. Algebraic methods and computational results

In this section, we prove Theorem 3 and Theorem 4. The following is Theorem 3 from the introduction.

Theorem 23. *Let $n = kp^\alpha$ be an odd integer where p is a prime, $\alpha \geq 1$ and r is given. Define $a = \frac{k-1}{2}$, $c = \left\lceil \frac{r}{p^\alpha} \right\rceil$ and*

$$F_k(x) := \sum_{i=-a}^{a} \left[\prod_{j=-a}^{a} (x-j)\right] \frac{1}{x-i}.$$

Assume that $\nu_p(F_k(c+a)) \leq \nu_p(k!)$. Then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. In particular if $c+a$ is not a root of $F_k(x)$ in modulo p, then $h_n^{(r)} \notin \mathbb{N}$.

Proof. By Lemma 12, we know that $|I_{p^\alpha}(n,r)| = \frac{n}{p^\alpha} = k$. Consider

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1})$$
$$= \frac{\mathrm{N}}{\mathrm{D}} \cdot \frac{cp^\alpha \cdot (c+1)p^\alpha \cdots (c+k-1)p^\alpha}{p^\alpha \cdot 2p^\alpha \cdots kp^\alpha} \cdot \left(\sum_{i=0}^{k-1} \frac{1}{(c+i)p^\alpha} + \sum_{\substack{\ell=r \\ p^\alpha \nmid \ell}}^{n+r-1} \frac{1}{\ell} \right),$$

where $\nu_p(\mathrm{N}) = \nu_p(\mathrm{D})$. To see this fact, first note that N is the multiplication of the elements in $I(n,r)$ which are not divisible by p^α whereas D is the multiplication of the elements in $I(n,1)$ which are not divisible by p^α. Recall by Lemma 12 that for all $1 \le \beta < \alpha$ we have $\left|I_{p^\beta}(n,r)\right| = \frac{n}{p^\beta} = kp^{\alpha-\beta}$. This implies that the number of elements in the set

$$J_{p^\beta}(n,r) = \left\{a \in I(n,r) \,\middle|\, p^\beta \parallel a \right\} = \left\{a \in I(n,r) \,\middle|\, p^\beta \mid a \right\} \setminus \left\{a \in I(n,r) \,\middle|\, p^{\beta+1} \mid a \right\}$$

is equal to $kp^{\alpha-\beta} - kp^{\alpha-\beta-1}$. Since $\left|I_{p^\beta}(n,1)\right| = \left\lfloor \frac{n}{p^\beta} \right\rfloor = kp^{\alpha-\beta}$, we see that $\left|J_{p^\beta}(n,1)\right| = kp^{\alpha-\beta} - kp^{\alpha-\beta-1}$. Therefore for any $1 \le \beta < \alpha$, the number of terms which are exactly divisible by p^β in the multiplication of N and D are the same. Cancellation of these terms leads to the fact that $\nu_p(\mathrm{N}) = \nu_p(\mathrm{D})$. Now define $q_1 = \frac{\mathrm{N}}{\mathrm{D}}$,

$$q_2 = \sum_{\substack{\ell=r \\ p^\alpha \nmid \ell}}^{n+r-1} \frac{1}{\ell}$$

and $\mathrm{C} = c(c+1)\cdots(c+k-1)$. As we have shown earlier, we have $\nu_p(q_1) = 0$. Note that $\nu_p(\mathrm{C}) \ge 0$ since $\mathrm{C} \in \mathbb{Z}$. Moreover we have $\nu_p(q_2) > -\alpha$, as the lowest power of p in the sum of q_2 is $\alpha - 1$. Due to the non-Archimedian property of p-adic valuation we obtained the mentioned result. Observe that

$$h_n^{(r)} = \frac{q_1}{p^\alpha \cdot k!} \cdot \sum_{i=0}^{k-1} \left[\left(\prod_{j=0}^{k-1} (c+j) \right) \cdot \frac{1}{c+i} \right] + \frac{\mathrm{C} \cdot q_1 q_2}{k!}.$$

By the definition of $F_k(x)$ and a, we have

$$F_k(x+a) = \sum_{i=0}^{k-1} \left[\left(\prod_{j=0}^{k-1} (x+j) \right) \cdot \frac{1}{x+i} \right].$$

This implies that

$$\nu_p(h_n^{(r)}) \ge \min \left\{ \nu_p\left(\frac{q_1 \cdot F_k(c+a)}{p^\alpha \cdot k!} \right), \nu_p\left(\frac{\mathrm{C} \cdot q_1 q_2}{k!} \right) \right\}$$

by the non-Archimedean property of the p-adic valuation. Recall that $\nu_p(h_n^{(r)})$ is equal to either one of them unless $\nu_p\left(\frac{q_1\cdot F_k(c+a)}{p^\alpha\cdot k!}\right)=\nu_p\left(\frac{\mathrm{C}\cdot q_1q_2}{k!}\right)$.

Suppose that $\nu_p(F_k(c+a))\le\nu_p(k!)$. Then the p-adic valuation of the first term is

$$\nu_p\left(\frac{q_1\cdot F_k(c+a)}{p^\alpha\cdot k!}\right)=\nu_p(q_1)+\nu_p(F_k(c+a))-\nu_p(k!)-\nu_p(p^\alpha)\le-\alpha, \quad (4.1)$$

since $\nu_p(q_1)=0$. Due to the same reason, we get

$$\nu_p\left(\frac{\mathrm{C}\cdot q_1q_2}{k!}\right)=\nu_p(\mathrm{C})+\nu_p(q_2)-\nu_p(k!).$$

Observe that

$$\frac{\mathrm{C}}{k!}=\frac{c\cdot(c+1)\cdots(c+k-1)}{1\cdot 2\cdots k}=\binom{c+k-1}{k}\in\mathbb{Z}.$$

Therefore we have $\nu_p(\mathrm{C})\ge\nu_p(k!)$. This indicates that the p-adic valuation of the second term satisfies

$$\begin{aligned}\nu_p\left(\frac{\mathrm{C}\cdot q_1q_2}{k!}\right)&=\nu_p(\mathrm{C})+\nu_p(q_2)-\nu_p(k!)\ge\nu_p(q_2)\\&>-\alpha\ge\nu_p\left(\frac{q_1\cdot F_k(c+a)}{p^\alpha\cdot k!}\right),\end{aligned} \quad (4.2)$$

due to (4.1). Hence we obtain by the non-Archimedean property that $\nu_p(h_n^{(r)})$ is equal to the minimum of these two terms. By (4.2) we also deduce that

$$\nu_p(h_n^{(r)})=\nu_p\left(\frac{q_1\cdot F_k(c+a)}{p^\alpha\cdot k!}\right)\le-\alpha\le-1;$$

and thus $h_n^{(r)}\notin\mathbb{N}$.

For the second case, assume that $c+a$ is not a root of $F_k(x)$. This implies that $F_k(c+a)\not\equiv 0 \mod p$; and hence $\nu_p(F_k(c+a))=0$. Since $k!$ is an integer, we have $\nu_p(k!)\ge 0=\nu_p(F_k(c+a))$. Thus we conclude by the first part of the theorem that the corresponding $h_n^{(r)}$ is not an integer. □

Note that the last part of Theorem 18 can also be shown by using Theorem 23. To do so, take $k=1$. In that case $a=0$ and the polynomial $F_1(x)$ is the constant 1 polynomial. Thus for any integer c and $p\in\mathbb{P}$ we have $\nu_p(F_1(c))=0$. We also have $\nu_p(k!)=\nu_p(1)=0$, which implies that $\nu_p(F_k(c))\le\nu_p(k!)$. Since the condition given in Theorem 23 is satisfied, we conclude that $h_{p^\alpha}^{(r)}$ is not an integer for any $r\in\mathbb{N}$.

In order to show that $h_n^{(r)}$ is not an integer, it is enough to find one such prime p in the factorization of n that satisfies the properties given in Theorem 23. In other words, an integer hyperharmonic number $h_n^{(r)}$ has to satisfy the following property: for all p

that divides n and for all $\alpha \leq \nu_p(n)$, we have $F_k(c+a) \equiv 0 \mod p$ where $k = \frac{n}{p^\alpha}$, and the integers c and a are as in Theorem 23. Heuristically if n has ℓ many distinct prime divisors, say $n = {p_1}^{\alpha_1} \cdots {p_\ell}^{\alpha_\ell}$, then the probability that $F_k(c+a) \equiv 0 \mod p_i$ is $\frac{1}{p_i}$ for $i \in \{1, \ldots, \ell\}$, $k = \frac{n}{{p_i}^\beta}$ and $\beta \leq \alpha_i$. And hence the probability that $h_n^{(r)}$ is not an integer is $1 - \frac{1}{n}$. So it is very unlikely that $h_n^{(r)}$ is an integer, as it was proved in Theorem 1.

Obviously one can observe that if p is sufficiently large for a given r, i.e. $p > F_k(c+a)$, then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Recall that this situation was investigated in Example 22 and its generalization just after this example. Also note that, if $F_k(x)$ has no roots modulo the given prime p then again for any $r \in \mathbb{N}$ we deduce by Theorem 23 that $h_n^{(r)}$ is not an integer. This fact will be useful to prove the following corollary which is a part of Theorem 4:

Corollary 24. *Let $r, \alpha \geq 1$ be any natural numbers. If p is a prime such that $p \equiv \pm 5 \mod 12$, then $h_{3p^\alpha}^{(r)} \notin \mathbb{N}$. Apart from that if*

$$\begin{aligned} p \equiv\ & 7, 11, 13, 14, 19, 21, 22, 23, 26, 28, 31, 33, 38, 39, 41, 42, 44, 46, 52, 53, 56, \quad \mod 145, \\ & 57, 61, 62, 63, 66, 67, 69, 76, 78, 79, 82, 83, 84, 88, 89, 92, 93, 99, 101, 103, \\ & 104, 106, 107, 112, 114, 117, 119, 122, 123, 124, 126, 131, 132, 134, 138 \end{aligned} \tag{4.3}$$

then $h_{5p^\alpha}^{(r)} \notin \mathbb{N}$.

Proof. In order to show this corollary, we will prove that the corresponding polynomial $F_k(x)$ for $k = 3, 5$ has no roots in modulo p. For the first part of the corollary, note that $F_3(x) = 3x^2 - 1$. So $F_3(x)$ has no roots in modulo p if and only if

$$3x^2 - 1 \not\equiv 0 \mod p \Leftrightarrow x^2 \not\equiv 3^{-1} \mod p,$$

where 3^{-1} denotes the multiplicative inverse of 3 in $\mathbb{F}_p^\times$. This indicates that $F_3(x)$ has no roots in $\{0, \ldots, p-1\}$ if and only if 3^{-1} is not a square modulo p, which means that 3 is not a square modulo p. So in order to use Theorem 23, it is enough to have $\left(\frac{3}{p}\right) = -1$ where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. By the well-known quadratic reciprocity law (see [16, Proposition II.2.5]), we know that

$$\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}} \cdot \left(\frac{p}{3}\right). \tag{4.4}$$

If $p \equiv 5 \mod 12$, then $(-1)^{\frac{p-1}{2}} = 1$ and $\left(\frac{p}{3}\right) = \left(\frac{2}{3}\right) = -1$. So by (4.4), we deduce that $\left(\frac{3}{p}\right) = -1$, and hence $h_{3p^\alpha}^{(r)} \notin \mathbb{N}$. Similarly for a prime $p \equiv 7 \mod 12$, we have $(-1)^{\frac{p-1}{2}} = -1$ and $\left(\frac{p}{3}\right) = \left(\frac{1}{3}\right) = 1$. Again by (4.4), we deduce that $h_{3p^\alpha}^{(r)} \notin \mathbb{N}$.

For the second case, observe that $F_5(x) = 5x^4 - 15x^2 + 4 = g(y)$ where $g(y) = 5y^2 - 15y + 4$ and $y = x^2$. Note that $F_5(x)$ has no roots in $\mathbb{F}_p = \{0, \ldots, p-1\}$, if the discriminant Δ of $g(y)$ is not a square. Since $\Delta = 145 = 5 \cdot 29$, we obtain that $\left(\frac{145}{p}\right) = \left(\frac{5}{p}\right) \cdot \left(\frac{29}{p}\right)$. In order to get 145 is not a square modulo p, it is enough to have $\left(\frac{5}{p}\right) = -\left(\frac{29}{p}\right)$. Recall that by the quadratic reciprocity law, we have

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \cdot \left(\frac{p}{q}\right), \tag{4.5}$$

where q is either 5 or 29. Since $q \equiv 1 \bmod 4$ for $q = 5, 29$, we deduce by (4.5) that $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$. So we have two different cases: either

$$\left(\frac{p}{5}\right) = -\left(\frac{p}{29}\right) = 1 \quad \text{or} \quad \left(\frac{p}{5}\right) = -\left(\frac{p}{29}\right) = -1$$

holds. Note that each of these cases leads to a relationship between modulo 5 and 29. By using the list of squares in modulo 5, 29 and the well-known Chinese remainder theorem, one can obtain the modular equivalences given in (4.3). □

Observe that the modular equivalences given in Corollary 24 cannot be obtained by using combinatorial methods.

Remark 25. We can also compute $F_k(x)$ iteratively. To obtain that one can define the polynomial

$$G_k(x) := \prod_{i=-a}^{a} (x - i),$$

where $a = \frac{k-1}{2}$. According to this definition $G_k(x)$ is nothing but $x \prod_{i=1}^{a}(x^2 - i^2)$. By using this, we can deduce that

$$F_k(x) = \sum_{i=-a}^{a} \frac{G_k(x)}{x - i}.$$

Therefore we have

$$G_k(x) = (x^2 - a^2) \cdot G_{k-2}(x),$$
$$F_k(x) = (x^2 - a^2) \cdot F_{k-2}(x) + 2xG_{k-2}(x).$$

These equalities lead to the computation of $F_k(x)$ for the cases $k = 7, 9$:

$$F_7(x) = 7x^6 - 70x^4 + 147x^2 - 36,$$
$$F_9(x) = 9x^8 - 210x^6 + 1365x^4 - 2460x^2 + 576.$$

Table 1
The list of the values of x_d, n_d and $r_d = d \cdot n_d$, for a given $d \leq 16$.

d	x_d	n_d	r_d
1	53	53	53
2	127	127	254
3	127	164	492
4	149	192	768
5	149	192	960
6	223	287	1.722
7	223	287	2.009
8	307	395	3.160
9	331	426	3.834
10	331	426	4.260
11	331	426	4.686
12	541	696	8.352
13	541	696	9.048
14	541	696	9.744
15	541	696	10.440
16	541	696	11.136

For those cases we have to solve cubic and quartic reciprocities in the corresponding field $\mathbb{F}_p$, respectively. This requires a lot of computational cases and we will not elaborate it in this paper.

Now we can combine what we have achieved so far and the following result is the rest of Theorem 4:

Corollary 26. *Let $1 < n \leq 32$ be any integer. Then for all $r \in \mathbb{N}$ we have $h_n^{(r)} \notin \mathbb{N}$. Moreover for all $n > 1$ and $r \in \mathbb{N}$, if $r \leq 16n$ then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. In fact for all $n > 1$, if $r \leq 20.001$ then $h_n^{(r)}$ is not an integer.*

Proof. To see the first part of the corollary recall that if n is even or a prime power, then $h_n^{(r)}$ is not an integer due to the Theorem 18. The only odd natural numbers which are less than 33 and not prime powers are 15 and 21. Note that these numbers are of the form $3p$ where $p \equiv \pm 5 \bmod 12$. By Corollary 24 we deduce that $h_{15}^{(r)}$ and $h_{21}^{(r)}$ are not integers also for any $r \in \mathbb{N}$. Hence the first part of the corollary follows immediately.

We use Theorem 10 to prove the second part, i.e. for a given d we find the possible smallest n_d such that for all $n \geq n_d$ and $r \leq dn$ we have $h_n^{(r)} \notin \mathbb{N}$. To do so, we first calculate the corresponding lower bound x_d such that for all $x \geq x_d$ there is a prime lying in the interval $\left(\left(1 - \frac{1}{2d+7}\right) x, x\right]$. In order to find these x_d's, we use the bounds given in [13, Theorem 1.9]. After that for each $d \leq 16$, we compute the corresponding value of n_d where $n_d = \left\lceil \frac{9}{7} \cdot x_d \right\rceil$. According to our computations, we can decrease the value of n_d for $d = 1, 2$. In fact, n_d can be taken as 53 and 127 for $d = 1$ and $d = 2$, respectively. Table 1 shows the list of values of n_d for given d's up to 16.

According to this table, it is enough to check the lattice points in $\mathbb{R}^2$ where $n \leq 696$, $r \leq 11.136$. By using Theorem 18 and Corollary 24, we eliminate the values of n where

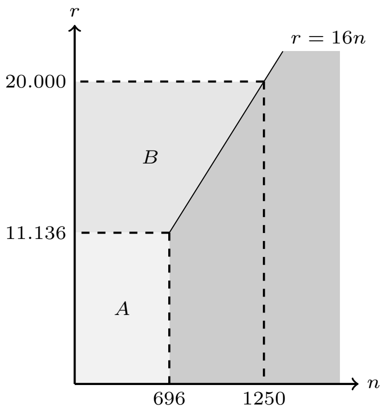

Fig. 2. The regions A and B contain the lattice points where we check for the non-integerness property of the corresponding hyperharmonic number $h_n^{(r)}$. The gray region shows the area that can be ignored due to Theorem 10 and Table 1.

$h_n^{(r)} \notin \mathbb{N}$. Thanks to [20], we observed that there are 165 possible values of $n \leq 696$ for which $h_n^{(r)}$ may be an integer. Due to Theorem 18 we can also ignore the odd values of r, while considering the region $A = \{(n,r) \in \mathbb{Z}^2 \,|1 < n \leq 696,\, 1 \leq r \leq 11.136\} \subseteq \mathbb{Z}^2$. By computer check, we found that $h_n^{(r)}$ is not an integer for $(n,r) \in A$.

For the last part of the corollary, observe that it is enough to check the non-integerness property of $h_n^{(r)}$ where the (n,r) tuples lie in the region

$$B = \{(n,r) \in \mathbb{Z}^2 \,|1 < n \leq 1250,\, 11.136 < r \leq 20.000,\, r > 16n\} \subseteq \mathbb{Z}^2.$$

The Fig. 2 shows the corresponding region.

Similar to the previous case, we eliminate the possible tuples $(n,r) \in B$ by using Theorem 18 and Corollary 24. Then by computer check again, we observed that for all $n > 1$ and $r \leq 20.000$ the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Note that if $r = 20.001$ then $h_n^{(r)} \notin \mathbb{N}$, due to Theorem 18. □

5. Concluding remarks

As it can be seen from Corollary 26 we can show that $h_n^{(r)}$ is never an integer for any $r \leq 20.001$. Note that this result is independent from the choice of n. However a similar bound for n is relatively small when it is compared to the case of r. In fact, the best possible bound that we can obtain for n is 32 as it is given in the same corollary. Even for $n = 33$, it cannot be shown by using the given methods that $h_n^{(r)}$ is not an integer for any $r \geq 1$. There are two reasons for that: first of all n is of the form $3p$ where $p \not\equiv \pm 5 \bmod 12$. Furthermore we have

$$F_3(x+1) = 3x^2 + 6x + 2 = 3(x-1)(x-8)$$

in $\mathbb{F}_{11}$. Therefore we cannot use Corollary 24. Secondly we see that $n = 3k$ where $k = 11$. According to the definition of $F_k(x)$ we have

$$F_{11}(x+5) = 11x^{10} + 550x^9 + 11880x^8 + 145200x^7 + 1104411x^6 + 5412330x^5 + 17084650x^4 + 33638000x^3 + 38260728x^2 + 21257280x + 3628800.$$

Thus the polynomial $F_{11}(x+5) \equiv 2x^{10} + x^9 + x^4 + 2x^3 \bmod 3$, moreover we have

$$F_{11}(x+5) = 2x^3(x-1)^4(x+1)^3$$

in $\mathbb{F}_3$. Observe that each of the elements in $\mathbb{F}_3$ is a root of this polynomial. Now we will construct an integer r where we cannot decide whether $h_{33}^{(r)}$ is an integer with our methods and also it is impossible to check computationally. First note that $33 \equiv 1 \bmod 4$. By the Chinese remainder theorem, consider the following integer r which satisfies the following three conditions:

- $r \equiv 0 \bmod 4$,
- $r \equiv 28 \bmod 81$,
- $r \equiv 1 \bmod 121$.

If we choose this r very large then we cannot apply analytic methods. Also the combinatorial method is not helpful as r is even and it does not suit to the conditions of Remarks 19 and 20. Moreover our algebraic method also does not work since we do not have the valuation bound in Theorem 23. As r can be very large as we want, computationally testing does not also work. Nevertheless we can give partial results for a large class of integers r for the case $n = 33$.

Our methods are strong enough to find the non-integer hyperharmonic numbers whose n is either even or a prime power. However as we have seen so far, the non-integerness property for a semi-prime n cannot be shown by using either one of these methods except the algebraic one. The latter method might be used to cover some semi-prime n's which are also independent from the choice of r. Moreover for any $n > 1$ by combining all of our methods and exploiting the prime factorization of n, we can give a huge class of integers r where the corresponding hyperharmonic number is not integer.

Usually we deal with the cases where $h_n^{(r)}$ is not an integer. However there might be some hyperharmonic numbers which are also integers. Recall that we use the primes which satisfy the main condition that is given in Proposition 6, in order to show that the corresponding hyperharmonic integer $h_n^{(r)}$ is not an integer. Observe that if $p \in \left(\frac{n}{2}, n\right] \cap \mathbb{P}_{\geq 3}$ and $|I_p(n,r)| = 2$, then $\nu_p(h_n^{(r)}) \geq 0$. In fact if $|I_p(n,1)| = k < p$ and $|I_p(n,r)| = k+1$, we have

$$h_n^{(r)} = \frac{\mathrm{N}_1}{\mathrm{D}_1} \cdot \frac{cp \cdot (c+1)p \cdots (c+k)p}{p \cdot 2p \cdots kp} \cdot \left(\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n+r-1}\right)$$

$$= \frac{\mathrm{N}_1}{\mathrm{D}_1} \cdot \frac{p^{k+1} \cdot c \cdot (c+1) \cdots (c+k)}{p^k \cdot k!} \cdot \left(\frac{1}{p} \cdot \left(\frac{1}{c} + \frac{1}{c+1} + \cdots + \frac{1}{c+k} \right) + \frac{\mathrm{N}_2}{\mathrm{D}_2} \right),$$

where $\nu_p(\mathrm{N}_1) = \nu_p(\mathrm{D}_1) = \nu_p(\mathrm{D}_2) = 0 \leq \nu_p(\mathrm{N}_2)$. This implies that

$$\begin{aligned} h_n^{(r)} &= \frac{\mathrm{N}_1 \cdot p}{\mathrm{D}_1 \cdot k!} \cdot c \cdot (c+1) \cdots (c+k) \cdot \left(\frac{1}{p} \cdot \frac{\mathrm{A}}{c \cdot (c+1) \cdots (c+k)} + \frac{\mathrm{N}_2}{\mathrm{D}_2} \right) \\ &= \frac{\mathrm{N}_1 \cdot \mathrm{A}}{\mathrm{D}_1 \cdot k!} + \frac{\mathrm{N}_1 \cdot \mathrm{N}_2}{\mathrm{D}_1 \cdot \mathrm{D}_2} \cdot \frac{p \cdot c \cdot (c+1) \cdots (c+k)}{k!}, \end{aligned}$$

where

$$\nu_p(\mathrm{A}) = \nu_p \left(\sum_{i=0}^{k} \frac{c \cdot (c+1) \cdots (c+k)}{c+i} \right) \geq 0.$$

Since $k < p$, we have $\nu_p(k!) = 0$. Therefore $\nu_p \left(\frac{\mathrm{N}_1 \cdot \mathrm{A}}{\mathrm{D}_1 \cdot k!} \right) = \nu_p(\mathrm{A}) \geq 0$. For the second summand, we also have

$$\nu_p \left(\frac{\mathrm{N}_1 \cdot \mathrm{N}_2}{\mathrm{D}_1 \cdot \mathrm{D}_2} \cdot \frac{p \cdot c \cdot (c+1) \cdots (c+k)}{k!} \right) = \nu_p(\mathrm{N}_1) + 1 + \nu_p(c \cdot (c+1) \cdots (c+k)) \geq 1.$$

By the non-Archimedian property of the p-adic valuation, we deduce that

$$\nu_p(h_n^{(r)}) \geq \min \left\{ \nu_p \left(\frac{\mathrm{N}_1 \cdot \mathrm{A}}{\mathrm{D}_1 \cdot k!} \right), \nu_p \left(\frac{\mathrm{N}_1 \cdot \mathrm{N}_2}{\mathrm{D}_1 \cdot \mathrm{D}_2} \cdot \frac{p \cdot c \cdot (c+1) \cdots (c+k)}{k!} \right) \right\} \geq 0.$$

Note that for any $p \mid n$ and $r \geq 1$, we have $|I_p(n,r)| = \frac{n}{p}$ due to Lemma 12. So if there is an integer hyperharmonic number $h_n^{(r)}$, then one may expect that $|I_p(n,r)| = \left\lfloor \frac{n}{p} \right\rfloor + 1$ for some $p \nmid n$, $p < n$ and $\left\lfloor \frac{n}{p} \right\rfloor < p$. This can be a key idea to construct a hyperharmonic number which is also an integer, if such a number exists.

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