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RESEARCH ARTICLE

Generating sets of finite singular transformation semigroups

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Dedicated to the memory of John Mackintosh Howie (1936–2011).

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Abstract J.M. Howie proved that Sing_n, the semigroup of all singular mappings of $\{1,\dots,n\}$ into itself, is generated by its idempotents of defect 1 (in J. London Math. Soc. 41, 707–716, 1966). He also proved that if $n \geq 3$ then a minimal generating set for Sing_n contains $n(n-1)/2$ transformations of defect 1 (in Gomes and Howie, Math. Proc. Camb. Philos. Soc. 101. 395–403, 1987). In this paper we find necessary and sufficient conditions for any set for transformations of defect 1 in Sing_n to be a (minimal) generating set for Sing_n.

Keywords Singular transformation semigroups · Idempotents · (Minimal) generating set

1 Introduction

The full transformation semigroup $\mathcal{T}_X$ on a set X and the semigroup analogue of the symmetric group $\mathcal{S}_X$ has been much studied over the last fifty years, in both the finite and the infinite cases. Among recent contributions are [1–3, 8]. Here we are concerned solely with the case where $X = X_n = \{1,\dots,n\}$, and we write respectively $\mathcal{T}_n$ and $\mathcal{S}_n$ rather than $\mathcal{T}_{X_n}$ and $\mathcal{S}_{X_n}$. Moreover, the semigroup $\mathcal{T}_n \setminus \mathcal{S}_n$ of all singular self-maps of X_n is denoted by Sing_n. (For unexplained terms in semigroup theory, see [7]). Let S be any semigroup, and let A be any nonempty subset of S. Then the subsemigroup generated by A, that is the smallest subsemigroup of S containing A, denoted by $\langle A\rangle$. It is well known that the *rank* of Sing_n, defined by

$$\mathrm{rank}(\mathrm{Sing}_n) = \min\{|A| : \langle A\rangle = \mathrm{Sing}_n\},$$

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is equal to $n(n-1)/2$ for $n \geq 3$ (see [4]). The *idempotent rank*, defined as the cardinality of a minimal generating set of idempotents, of Sing_n is also equal to $n(n-1)/2$ (see [6]). Recently, it has been proved in [3] that, for $2 \leq r \leq m \leq n$, the (m, r)-*rank*, defined as the cardinality of a minimal generating set of (m, r)-path-cycles, of Sing_n is once again $n(n-1)/2$. The main goal of this study is to investigate whether a given subset A of transformations of defect 1 in Sing_n is a (minimal) generating set for Sing_n.

Let $\alpha \in T_n$. The *Defect set*, *defect*, *kernel* and *Fix* of α are defined by

$$\begin{aligned}
\mathrm{Def}(\alpha) &= X_n \setminus \mathrm{im}(\alpha) \\
\mathrm{def}(\alpha) &= \left|\mathrm{Def}(\alpha)\right| \\
\ker(\alpha) &= \left\{(x, y) \in X_n \times X_n : x\alpha = y\alpha\right\} \\
\mathrm{Fix}(\alpha) &= \{x \in X_n : x\alpha = x\},
\end{aligned}$$

respectively. For any $\alpha, \beta \in T_n$, it is well known that $\ker(\alpha) \subseteq \ker(\alpha\beta)$, $\mathrm{im}(\alpha\beta) \subseteq \mathrm{im}(\beta)$, and that

$$\begin{aligned}
(\alpha, \beta) \in \mathcal{D} \quad &\Leftrightarrow \quad \left|\mathrm{im}(\alpha)\right| = \left|\mathrm{im}(\beta)\right| \quad \Leftrightarrow \quad \mathrm{def}(\alpha) = \mathrm{def}(\beta) \\
(\alpha, \beta) \in \mathcal{H} \quad &\Leftrightarrow \quad \ker(\alpha) = \ker(\beta) \text{ and } \mathrm{im}(\alpha) = \mathrm{im}(\beta).
\end{aligned}$$

We denote the $\mathcal{D}$-Green class of all singular self maps of defect r by D_{n-r} $(1 \leq r \leq n-1)$. It is clear that $\alpha \in D_{n-1}$ if and only if there exist $i, j \in X_n$ with $i \neq j$ such that $\ker(\alpha)$ is the equivalence relation on X_n generated by $\{(i, j)\}$, or equivalently, generated by $\{(j, i)\}$. In this case we define the set $\mathrm{Ker}(\alpha)$ by

$$\mathrm{Ker}(\alpha) = \{i, j\}.$$

(Notice that $\ker(\alpha)$ denotes an equivalence relation on X_n, and that $\mathrm{Ker}(\alpha)$ denotes a subset of X_n.) Equivalently, $\alpha \in D_{n-1}$ if and only if there exists $k \in X_n$ such that $\mathrm{Def}(\alpha) = \{k\}$. It is well known that $\alpha \in T_n$ is an idempotent element if and only if the restriction of α to $\mathrm{im}(\alpha)$ is the identity map on $\mathrm{im}(\alpha)$. We denote the set of all idempotents in a subset A of T_n by $E(A)$. Let α be in $E(D_{n-1})$ with $\mathrm{Def}(\alpha) = \{i\}$. Then there exists unique $j \in X_n$ such that $i \neq j$ and $i\alpha = j$. In this case we write

$$\alpha = \zeta_{i,j} = \begin{pmatrix} i \\ j \end{pmatrix}.$$

A digraph Π is called *complete* if, for all pairs of vertices $u \neq v$ in $V(\Pi)$, either the directed edge $(u, v) \in E(\Pi)$ or the directed edge $(v, u) \in E(\Pi)$. For two vertices $u, v \in V(\Pi)$, if there exist a directed path from u to v then we say u is *connected to* v in Π. We say Π is *strongly connected* if, for any two vertices $u, v \in V(\Pi)$, u is connected to v in Π.

Let A be a subset of $E(D_{n-1}) = \{\zeta_{i,j} : i, j \in X_n \text{ and } i \neq j\}$. Then we define the digraph Δ_A as follows:

- the vertex set of Δ_A, denoted by $V = V(\Delta_A)$, is X_n; and
- the directed edge set of Δ_A, denoted by $\vec{E} = \vec{E}(\Delta_A)$, is

$$\vec{E} = \{(j,i) \in V \times V : \zeta_{i,j} \in A\}.$$

For any subset A of $E(D_{n-1})$ it is shown in [6] that A is a minimal generating set for Sing_n if and only if the digraph Δ_A is complete and strongly connected.

Now let A be a nonempty subset of D_{n-1}. Then we define another digraph Γ_A as follows:

- the vertex set of Γ_A, denoted by $V = V(\Gamma_A)$, is A; and
- the directed edge set of Γ_A, denoted by $\vec{E} = \vec{E}(\Gamma_A)$, is

$$\vec{E} = \{(\alpha,\beta) \in V \times V : \mathrm{Def}(\alpha) \subseteq \mathrm{Ker}(\beta)\}.$$

The main goal of this paper is to prove that any subset A of D_{n-1} is a (minimal) generating set for Sing_n if and only if, for each idempotent $\zeta_{i,j} \in E(D_{n-1})$, there exist $\alpha, \beta \in A$ such that

(i) $\mathrm{Ker}(\alpha) = \{i, j\}$,
(ii) $\mathrm{Def}(\beta) = \{i\}$, and
(iii) α is connected to β in the digraph Γ_A.

Moreover, for $A \subseteq E(D_{n-1})$, we show that the directed graph Δ_A is complete and strongly connected if and only if, for each idempotent $\zeta_{i,j} \in E(D_{n-1})$, there exist $\alpha, \beta \in A$ such that $\mathrm{Ker}(\alpha) = \{i, j\}$, $\mathrm{Def}(\beta) = \{i\}$ and that α is connected to β in the digraph Γ_A. Therefore, the main result of this paper includes the result of [6, Theorem 1].

2 Preliminaries

For an $\alpha \in D_{n-1}$ with $\mathrm{Ker}(\alpha) = \{i, j\}$ and $\mathrm{Def}(\alpha) = \{k\}$ we define the idempotents α^e and α^f, and the permutation α^p on X_n as follows:

First we define α^e as

$$\text{either} \quad \alpha^e = \begin{pmatrix} i \\ j \end{pmatrix} \quad \text{or} \quad \alpha^e = \begin{pmatrix} j \\ i \end{pmatrix}.$$

In the first case we define

$$\alpha^f = \begin{pmatrix} k \\ i\alpha \end{pmatrix} \quad \text{and} \quad x\alpha^p = \begin{cases} k & x = i \\ x\alpha & x \neq i, \end{cases}$$

and in the second case we define

$$\alpha^f = \begin{pmatrix} k \\ j\alpha \end{pmatrix} \quad \text{and} \quad x\alpha^p = \begin{cases} k & x = j \\ x\alpha & x \neq j. \end{cases}$$

Notice that in both cases

$$\alpha^f = \binom{k}{i\alpha} = \binom{k}{j\alpha},$$

and we have

$$\alpha = \alpha^e \alpha^p = \alpha^p \alpha^f. \tag{1}$$

For example, if $\alpha = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 2 \end{smallmatrix}\right)$ then, since $\mathrm{Ker}(\alpha) = \{1, 2\}$ and $\mathrm{Def}(\alpha) = \{1\}$, we have

$$\alpha = \alpha^e \alpha^p = \binom{1}{2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \binom{1}{3} = \alpha^p \alpha^f$$

or

$$\alpha = \alpha^e \alpha^p = \binom{2}{1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \binom{1}{3} = \alpha^p \alpha^f.$$

It is well known that each element in Sing_n can be written as a product of idempotents from $E(D_{n-1})$. Now we state a very easy lemma which will be useful throughout this paper.

Lemma 1 *For $i \neq j, k$ we have $\binom{i}{j}\binom{i}{k} = \binom{i}{j}$.*

Lemma 2 *For $n \geq 3$ let $\alpha, \beta \in D_{n-1}$. Then $\alpha\beta \in D_{n-1}$ if and only if* $\mathrm{Def}(\alpha) \subseteq \mathrm{Ker}(\beta)$.

Proof With the notation in (1), since $\alpha\beta = (\alpha^p \alpha^f)(\beta^e \beta^p)$, it follows that $\alpha\beta \in D_{n-1}$ if and only if $\alpha^f \beta^e \in D_{n-1}$.

($\Rightarrow$) Suppose that $\alpha\beta \in D_{n-1}$, and that $\mathrm{Def}(\alpha) = \{i\}$ and $\mathrm{Ker}(\beta) = \{k, l\}$. Then either $\beta^e = \binom{k}{l}$ or $\beta^e = \binom{l}{k}$, and there exists $i \neq j \in X_n$ such that $\alpha^f = \binom{i}{j}$. In both cases i must be equal to either k or l since $\alpha^f \beta^e \in D_{n-1}$.

($\Leftarrow$) Suppose that $\mathrm{Def}(\alpha) = \{i\} \subseteq \mathrm{Ker}(\beta)$. Then there exists $i \neq k \in X_n$ such that $\mathrm{Ker}(\beta) = \{i, k\}$. Since we can take $\beta^e = \binom{i}{k}$, it follows from Lemma 1 and (1) that

$$\begin{aligned} \alpha\beta &= (\alpha^p \alpha^f)(\beta^e \beta^p) = \alpha^p (\alpha^f \beta^e) \beta^p \\ &= (\alpha^p \alpha^f) \beta^p = (\alpha^e \alpha^p) \beta^p \\ &= \alpha^e (\alpha^p \beta^p). \end{aligned}$$

Since $\alpha^e \in E(D_{n-1})$ and $\alpha^p \beta^p \in S_n$, it follows that $\alpha\beta = \alpha^e(\alpha^p \beta^p) \in D_{n-1}$, as required. □

Denote the $\mathcal{H}$-Green class containing the idempotent $\binom{i}{j}$ in D_{n-1} by $H_{i,j}$. Then it is clear that $\alpha \in H_{i,j}$ if and only if $\mathrm{Def}(\alpha) = \{i\}$ and $\mathrm{Ker}(\alpha) = \{i, j\}$.

Lemma 3 *Let $\alpha \in D_{n-1}$. Then $\alpha \in H_{i,j}$ if and only if there exists a permutation $\beta \in S_n$ such that $\alpha = \binom{i}{j}\beta$ and $i \in \mathrm{Fix}(\beta)$.*

Proof (⇒) Suppose that $\alpha \in H_{i,j}$. Since $\mathrm{Def}(\alpha) = \{i\}$ and $\mathrm{Ker}(\alpha) = \{i, j\}$, we can take $\alpha^e = \binom{i}{j}$. Moreover, it follows from the definition of α^p that $i \in \mathrm{Fix}(\alpha^p)$. Thus, from (1), we have $\alpha = \alpha^e \alpha^p$, as required.

(⇐) Suppose that there exists a permutation $\beta \in S_n$ such that $\alpha = \binom{i}{j}\beta$ and $i \in \mathrm{Fix}(\beta)$. Since $\beta \in S_n$ and $i \in \mathrm{Fix}(\beta)$, it is clear that

$$\mathrm{im}(\alpha) = \mathrm{im}\left(\binom{i}{j}\right) = X_n \setminus \{i\},$$

and that $\mathrm{Ker}(\alpha) = \{i, j\}$. Hence, $\alpha \in H_{i,j}$, as required. □

3 Generating Set

Theorem 4 *Let A be a subset of the $\mathcal{D}$-Green class D_{n-1}. Then A is a generating set of* Sing_n *if and only if, for each idempotent $\binom{i}{j} \in E(D_{n-1})$, there exist $\alpha, \beta \in A$ such that*

(i) $\mathrm{Ker}(\alpha) = \{i, j\}$,
(ii) $\mathrm{Def}(\beta) = \{i\}$, *and*
(iii) *α is connected to β in the digraph Γ_A.*

Proof (⇒) Suppose that A is a generating set for Sing_n. Then, for each $\zeta_{i,j} = \binom{i}{j} \in E(D_{n-1})$, there exist $\alpha_1, \ldots, \alpha_r \in A$ such that

$$\alpha_1 \cdots \alpha_r = \zeta_{i,j}.$$

Since $\alpha_1, \ldots, \alpha_r, \zeta_{i,j} \in D_{n-1}$, $\ker(\alpha_1) \subseteq \ker(\zeta_{i,j})$ and $\mathrm{im}(\zeta_{i,j}) \subseteq \mathrm{im}(\alpha_r)$, it follows that

$$\mathrm{Ker}(\alpha_1) = \mathrm{Ker}(\zeta_{i,j}) = \{i, j\},$$

$$\mathrm{Def}(\alpha_r) = \mathrm{Def}(\zeta_{i,j}) = X_n \setminus \mathrm{im}(\zeta_{i,j}) = \{i\}$$

and that, for each $1 \le k \le r-1$, we have $\alpha_k \alpha_{k+1} \in D_{n-1}$. Then, from Lemma 2, we have $\mathrm{Def}(\alpha_k) \subseteq \mathrm{Ker}(\alpha_{k+1})$. Hence there exists a directed path from α_1 to α_r, that is α_1 is connected to α_r in Γ_A, as required.

(⇐) Since each element in Sing_n can be written as a product of idempotents in D_{n-1}, it is enough to show that $E(D_{n-1}) \subseteq \langle A \rangle$.

Let $\zeta_{i,j} \in E(D_{n-1})$. From (i) and (ii) there exist $\alpha, \beta \in A$ such that $\mathrm{Ker}(\alpha) = \{i, j\}$ and $\mathrm{Def}(\beta) = \{i\}$. Then it follows from (iii) that there exists a directed path from α to β, say

$$\alpha = \alpha_1 \to \alpha_2 \to \cdots \to \alpha_{r-1} \to \alpha_r = \beta$$

such that, for each $1 \le t \le r-1$,

$$\mathrm{Def}(\alpha_t) \subseteq \mathrm{Ker}(\alpha_{t+1}).$$

Since $\mathrm{Def}(\alpha_t) \subseteq \mathrm{Ker}(\alpha_{t+1})$ for each $1 \leq t \leq r-1$, it follows from Lemma 2 inductively that $\alpha_1 \cdots \alpha_r \in D_{n-1}$, and so $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\alpha_1 \cdots \alpha_r)$ and $\mathrm{im}(\beta) = \mathrm{im}(\alpha_1 \cdots \alpha_r)$. Therefore, $\alpha_1 \cdots \alpha_r \in H_{i,j}$. Since $H_{i,j}$ is a finite group with the identity $\zeta_{i,j}$, it follows that there exists a positive integer m such that $\zeta_{i,j} = (\alpha_1 \cdots \alpha_r)^m \in \langle A \rangle$, as required. □

Since $\mathrm{rank}(\mathrm{Sing}_n) = \frac{n(n-1)}{2}$, a generating set of Sing_n with $\frac{n(n-1)}{2}$ elements is a minimal generating set. Thus we have the following corollary from Theorem 4:

Corollary 5 *Let A be a subset of D_{n-1} with $\frac{n(n-1)}{2}$ elements. Then A is a minimal generating set of* Sing_n *if and only if, for each idempotent $\binom{i}{j} \in E(D_{n-1})$, there exist $\alpha, \beta \in A$ such that*

(i) $\mathrm{Ker}(\alpha) = \{i, j\}$,
(ii) $\mathrm{Def}(\beta) = \{i\}$, *and*
(iii) *α is connected to β in the digraph Γ_A.*

4 Remarks

From Theorem 4 and [6] we have the following result with the notation above:

Corollary 6 *Let A be a subset of $E(D_{n-1})$ with $\frac{n(n-1)}{2}$ elements. Then the directed graph Δ_A is complete and strongly connected if and only if, for each idempotent $\binom{i}{j} \in E(D_{n-1})$, there exist $\alpha, \beta \in A$ such that*

(i) $\mathrm{Ker}(\alpha) = \{i, j\}$,
(ii) $\mathrm{Def}(\beta) = \{i\}$, *and*
(iii) *α is connected to β in the digraph Γ_A.*

In other words, the main result of J.M. Howie in [6] coincides with our Theorem 4 when A is a subset of $E(D_{n-1})$ with $\frac{n(n-1)}{2}$ elements.

For example, let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ where

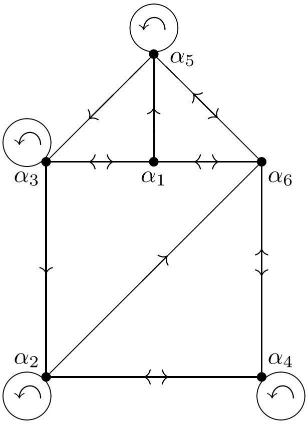
$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 2 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 4 & 2 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 2 \end{pmatrix}, \qquad \alpha_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 4 \end{pmatrix},$$

$$\alpha_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 3 \end{pmatrix}, \qquad \alpha_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 1 \end{pmatrix}.$$

With the notation above, for $\zeta_{1,2} \in E(D_{4-1})$, we have $\mathrm{Ker}(\alpha_1) = \{1, 2\}$ and $\mathrm{Def}(\alpha_3) = \{1\}$. Similarly, for all other elements of $E(D_{4-1})$, it can be shown that conditions (i) and (ii) of Theorem 4 are satisfied.

Moreover, Γ_A:



is a Hamiltonian digraph since the cycle

$$\alpha_5 \to \alpha_3 \to \alpha_2 \to \alpha_4 \to \alpha_6 \to \alpha_1 \to \alpha_5$$

is a Hamiltonian cycle. Hence, Γ_A is strongly connected, and so condition (iii) of Theorem 4 is satisfied. Therefore, A is a (minimal) generating set of Sing_4.

Indeed, since

$$\begin{array}{lll} \zeta_{1,2} = (\alpha_1\alpha_3)^3, & \zeta_{1,3} = \alpha_2\alpha_6\alpha_5\alpha_3, & \zeta_{1,4} = \alpha_3^3 \\ \zeta_{2,1} = \alpha_1\alpha_6, & \zeta_{2,3} = (\alpha_4\alpha_6)^2, & \zeta_{2,4} = (\alpha_5\alpha_6)^2 \\ \zeta_{3,1} = \alpha_2^3, & \zeta_{3,2} = \alpha_4^2, & \zeta_{3,4} = (\alpha_6\alpha_4)^2 \\ \zeta_{4,1} = (\alpha_3\alpha_1)^3, & \zeta_{4,2} = \alpha_5^3, & \zeta_{4,3} = (\alpha_6\alpha_5)^2, \end{array}$$

it follows that $E(D_{4-1}) \subseteq \langle A \rangle$, and so A is a (minimal) generating set for Sing_4. Notice that, to write each $\zeta_{i,j} \in E(D_{4-1})$ as a product of elements of A above, we use the paths in Γ_A. Therefore, Γ_A is also useful for writing the idempotents of defect 1 as a product of elements of A.

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